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A TREATISE
OF
PLANE TRIGONOMETRY.

TO WHICH IS PREFIXED,
A SUMMARY VIEW OF THE NATURE AND USE OF
LOGARITHMS.

BEING
THE SECOND PART
OF
A COURSE OF MATHEMATICS

ADAPTED TO THE METHOD OF INSTRUCTION
IN THE AMERICAN COLLEGES.

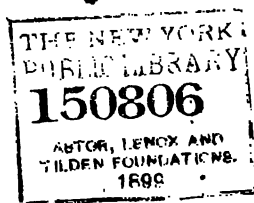
BY JEREMIAH DAY,
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Yale College.

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THE plan upon which this work was originally commenced, is continued in this second part of the course. As the single object is to provide for a *class in college*, such matter as is not embraced by this design is excluded; with the exception of a few things in the notes. The mode of treating the subjects, for the reasons mentioned in the preface to Algebra, is, in a considerable degree, diffuse. It was thought better to err on this extreme, than on the other, especially in the early part of the course. A more concise method may be adopted in the succeeding numbers.

The section on right angled triangles will probably be considered as needlessly minute. The solutions might, in all cases, be effected by the theorems which are given for oblique angled triangles. But the applications of rectangular trigonometry are so numerous, in navigation, surveying, astronomy, &c. that it was deemed important, to render familiar the various methods of stating the relations of the sides and angles; and especially to bring distinctly into view the principle on which most trigonometrical calculations are founded, the proportion between the parts of the given triangle, and a similar one formed from the sines, tangents, &c. in the tables.

The solutions of oblique angled triangles are made by the common methods. The propositions which are used in particular cases, principally by astronomers, are inserted in a note at the end. On the subject of *Trigonometrical Analysis*, nothing more than a few of the first principles could be admitted, in a work upon so limited a plan.

This number begins with a view of the nature and use of *Logarithms*, as preparatory to the calculations in Trigonometry.

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LOGARITHMS.

SECTION I.

NATURE OF LOGARITHMS.*

ART. 1. **T**HE operations of Multiplication and Division, when they are to be often repeated, become so laborious, that it is an object of importance to substitute, in their stead, more simple methods of calculation, such as Addition and Subtraction. If these can be made to perform, in an expeditious manner, the office of multiplication and division, a great portion of the time and labour which the latter processes require, may be saved.

Now it has been shown, (Algebra, 233, 237,) that *powers* may be multiplied, by adding their *exponents*, and divided, by subtracting their exponents. In the same manner, *roots* may be multiplied and divided, by adding and subtracting their fractional exponents. (Alg. 280, 286.) When these exponents are arranged in tables, and applied to the general purposes of calculation, they are called *Logarithms*.

2. LOGARITHMS, then, are the EXPONENTS of a series of powers and roots.†

In forming a system of logarithms, some particular number is fixed upon, as the *radix* or first power, whose logarithm is always 1. From this, a series of powers is raised, and the exponents of these are arranged in tables for use. To explain this, let the number which is chosen for the first power,

*Maskelyne's Preface to Taylor's Logarithms. Introduction to Hutton's Tables. Keil on Logarithms. Maseres Scriptores Logarithmici. Briggs' Logarithms. Dodson's Anti-logarithmic Canon. Euler's Algebra.

† See note A.

be represented by a . Then taking a series of powers, both direct and reciprocal, as in Alg. 207;

$$a^4, a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, a^{-4}, \&c.$$

The logarithm of a^3 is 3, And the logarithm of a^{-1} is -1 ,
 of a^1 is 1, of a^{-2} is -2 ,
 of a^0 is 0, of a^{-3} is -3 , &c.

Universally, the logarithm of a^x is x .

3. In the system of logarithms in common use, called *Briggs'* logarithms, the number which is taken for the radix is 10. The above series then, by substituting 10 for a , becomes

$$10^4, 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, \&c.$$

$$\text{Or } 10000, 1000, 100, 10, 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \&c.$$

Whose logarithms are

$$4, 3, 2, 1, 0, -1, -2, -3, \&c.$$

4. The fractional exponents of *roots*; and of powers of roots, are converted into *decimals*, before they are inserted in the logarithmic tables. See Alg. 255.

$$\begin{aligned} \text{The logarithm of } a^{\frac{1}{3}}, \text{ or } a^{0.3333}, &\text{ is } 0.3333, \\ \text{of } a^{\frac{2}{3}}, \text{ or } a^{0.6666}, &\text{ is } 0.6666, \\ \text{of } a^{\frac{3}{7}}, \text{ or } a^{0.4285}, &\text{ is } 0.4285, \\ \text{of } a^{\frac{11}{3}}, \text{ or } a^{3.6666}, &\text{ is } 3.6666, \&c. \end{aligned}$$

These decimals are carried to a greater or less number of places, according to the degree of accuracy required.

5. In forming a system of logarithms, it is necessary to obtain the logarithm of each of the numbers in the natural series 1, 2, 3, 4, 5, &c.; so that the logarithm of any number may be found in the tables. For this purpose, the *radix* of the system must first be determined upon; and then every other number may be considered as some power or root of this. If the radix is 10, as in the common system, every other number is to be considered as some power of 10.

That a power or root of 10 may be found, which shall be equal to any other number whatever, or, at least, a very near approximation to it, is evident from this, that the *exponent* may be endlessly varied; and if this be increased or diminished, the *power* will be increased or diminished.

If the exponent is a fraction, and the *numerator* be increased, the power will be increased: but if the *denominator* be increased, the power will be diminished.

6. To obtain then the logarithm of any number, according to Briggs' system, we have to find a power or root of 10 which shall be equal to the proposed number. The *exponent* of that power or root is the logarithm required. Thus

$$\left. \begin{array}{l} 7 = 10^{0.8451} \\ 20 = 10^{1.3010} \\ 30 = 10^{1.4771} \\ 400 = 10^{2.6020} \end{array} \right\} \text{therefore the } \left\{ \begin{array}{l} \text{of } 7 \text{ is } 0.8451 \\ \text{of } 20 \text{ is } 1.3010 \\ \text{of } 30 \text{ is } 1.4771 \\ \text{of } 400 \text{ is } 2.6020; \&c.* \end{array} \right.$$

7. A logarithm generally consists of two parts, an *integer* and a *decimal*. Thus the logarithm 2.60206, or, as it is sometimes written, 2+.60206, consists of the integer 2, and the decimal .60206. The integral part is called the *characteristic* or *index*† of the logarithm; and is frequently omitted, in the common tables, because it can be easily supplied, whenever the logarithm is to be used in calculation.

By art. 3d, the logarithms of

$$\begin{array}{l} 10000, 1000, 100, 10, 1, .1, .01, .001, \&c. \\ \text{are } 4, 3, 2, 1, 0, -1, -2, -3, \&c. \end{array}$$

As the logarithms of 1 and of 10 are 0 and 1, it is evident, that, if any given number be *between* 1 and 10, its logarithm will be between 0 and 1, that is, it will be greater than 0, but less than 1. It will therefore have 0 for its index, with a decimal annexed.

Thus the logarithm of 5 is 0.69697.

For the same reason, if the given number be between

$$\left. \begin{array}{l} 10 \text{ and } 100, \\ 100 \text{ and } 1000, \\ 1000 \text{ and } 10000, \end{array} \right\} \begin{array}{l} \text{the log.} \\ \text{will be} \\ \text{between} \end{array} \left\{ \begin{array}{l} 1 \text{ and } 2, \text{ i. e. } 1 + \text{the dec. part} \\ 2 \text{ and } 3, \quad 2 + \text{the dec. part} \\ 3 \text{ and } 4, \quad 3 + \text{the dec. part} \end{array} \right.$$

We have, therefore, when the logarithm of an integer or mixed number is to be found, this general rule,

*For the method of calculating Logarithms, see Fluxions.

† The term *index*, as it is used here, may possibly lead to some confusion in the mind of the learner. For the logarithm itself is the index or exponent of a power. The characteristic, therefore, is the index of an index.

8. *The index of the logarithm is always one less, than the number of integral figures, in the natural number whose logarithm is sought:* or, the index shows how far the first figure of the natural number is removed from the place of units.

Thus the logarithm of 37 is 1.56620.

Here, the number of figures being *two*, the index of the logarithm is 1.

The logarithm of 253 is 2.40312.

Here, the proposed number 253 consists of *three* figures, the first of which is in the *second* place from the unit figure. The index of the logarithm is therefore 2.

The logarithm of 62.8 is 1.79796.

Here it is evident that the mixed number 62.8 is between 10 and 100. The index of its logarithm must therefore be 1.

9. As the logarithm of 1 is 0, the logarithm of a number *less* than 1, that is, of any proper *fraction*, must be *negative*.

Thus by art. 3d

The logarithm of $\frac{1}{10}$ or .1 is -1 ,
 of $\frac{1}{100}$ or .01 is -2 ,
 of $\frac{1}{1000}$ or .001 is -3 , &c.

10. If the proposed number is *between* $\frac{1}{10}$ and $\frac{1}{100}$, its logarithm must be between -2 and -3 . To obtain the logarithm, therefore, we must either *subtract* a certain fractional part from -2 , or *add* a fractional part to -3 ; that is, we must either annex a *negative decimal* to -2 , or a *positive one* to -3 .

Thus the logarithm

of .005 is either $-2-.09691$, or $-3+.90309$.*

The latter is generally most convenient in practice, and is more commonly written $\overline{3}.90309$. The line over the index

* That these two expressions are of the same value will be evident, if we subtract the same quantity, $+.90309$ from each. The remainders will be equal, and therefore the quantities from which the subtraction is made must be equal.

From	$-2-.09691$	From	$-3+.90309$
Subtract	$+.90309$	Subtract	$+.90309$
Remainder	-3	Remainder	-3

See note B.

denotes, that *that* is negative, while the *decimal* part of the logarithm is positive.

The logarithm $\left\{ \begin{array}{l} \text{of } 0.3, \text{ is } \overline{1.47712}, \\ \text{of } 0.06, \text{ is } \overline{2.77815}, \\ \text{of } 0.009, \text{ is } \overline{3.95424}, \end{array} \right.$

And universally,

11. *The negative index of a logarithm shows how far the first significant figure of the natural number, is removed from the place of units, on the right; in the same manner as a positive index shows how far the first figure of the natural number is removed from the place of units, on the left. (Art. 8.)* Thus in the examples in the last article,

The decimal 3 is in the *first* place from that of units,
6 is in the *second* place,
9 is in the *third* place;

And the indices of the logarithms are $\overline{1}$, $\overline{2}$, and $\overline{3}$.

12. It is often more convenient, however, to make the *index* of the logarithm positive, as well as the decimal part. This is done by adding 10 to the index.

Thus, for -1 , 9 is written; for -2 , 8, &c.
Because $-1 + 10 = 9$, $-2 + 10 = 8$, &c.

With this alteration,

The logarithm $\left\{ \begin{array}{l} \overline{1.90309} \\ \overline{2.90309} \\ \overline{3.90309} \end{array} \right\}$ becomes $\left\{ \begin{array}{l} 9.90309, \\ 8.90309, \\ 7.90309, \text{ \&c.} \end{array} \right.$

This is making the index of the logarithm 10 too great. But with proper caution, it will lead to no error in practice.

13. The *sum* of the logarithms of two numbers, is the logarithm of the *product* of those numbers; and the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of one of the numbers divided by the other. (Art. 1.) In Briggs' system, the logarithm of 10 is 1. (Art. 3.) If therefore any number be multiplied or divided by 10, its logarithm will be increased or diminished by 1: and as this is an integer, it will only change the *index* of the logarithm, without affecting the decimal part.

Thus the logarithm of 4730 is 3.67486
 And the logarithm of 10 is 1.

The logarithm of the product 47300 is 4.67486
 And the logarithm of the quotient 473 is 2.67486

Here the *index* only is altered, while the decimal part remains the same. We have then this important property,

14. *The DECIMAL PART of the logarithm of any number is the same, as that of the number multiplied or divided by 10, 100, 1000, &c.*

Thus the log. of 45670,	is 4.65963,
4567,	3.65963,
456.7,	2.65963,
45.67,	1.65963,
4.567,	0.65963,
.4567,	1.65963, or 9.65963,
.04567,	2.65963, 8.65963,
.004567,	3.65963, 7.65963.

This property, which is peculiar to Briggs' system, is of great use in abridging the logarithmic tables. For when we have the logarithm of any number, we have only to change the *index*, to obtain the logarithm of every other number, whether integral, fractional, or mixed, consisting of the same significant figures. The decimal part of the logarithm of a fraction found in this way, is always *positive*. For it is the same as the decimal part of the logarithm of a whole number.

15. In a series of fractions *continually decreasing*, the *negative* indices of the logarithms *continually increase*. Thus

In the series 1, .1, .01, .001, .0001, .00001, &c.
 The logarithms are 0, -1, -2, -3, -4, -5, &c.

If the progression be continued, till the fraction is reduced to 0, the negative logarithm will become greater than any assignable quantity. The logarithm of 0, therefore, is *infinite and negative*. (Alg. 447.)

16. It is evident also, that all *negative* logarithms belong to fractions which are between 1 and 0; while *positive* logarithms belong to natural numbers which are greater than 1,

As the whole range of numbers, both positive and negative, is thus exhausted in supplying the logarithms of integral and fractional positive quantities; there can be no other numbers to furnish logarithms for *negative* quantities. On this account, the logarithm of a negative quantity is, by some writers, said to be *impossible*. It appears to be more proper, however, to consider the logarithms of negative quantities, as *being the same* with the logarithms of positive quantities. Logarithms are the exponents of powers and roots. (Art. 2.) But an exponent may be applied to a negative power or root, as well as to a positive one.

Thus the cube of $-a$ is $-a^3$ }
 And the cube of $+a$ is $+a^3$ } Alg. 218.

Though these two powers are one positive, and the other negative, yet their *exponents* are the same. So also, the logarithm of a , or of any power of a , is the same as the logarithm of the same power of $-a$. Positive and negative quantities, therefore, can not be distinguished from each other by their logarithms.

17. If a series of numbers be in GEOMETRICAL progression, their logarithms will be in ARITHMETICAL progression. For, in a geometrical series ascending, the quantities increase by a common *multiplier*; (Alg. 436.) that is, each succeeding term is the *product* of the preceding term into the ratio. But the *logarithm* of this product is the *sum* of the logarithms of the preceding term and the ratio; that is, the logarithms increase by a common *addition*, and are, therefore, in arithmetical progression. (Alg. 422.) In a geometrical progression *descending*, the terms decrease by a common *divisor*, and their logarithms, by a common *difference*.

Thus the numbers 1, 10, 100, 1000, 10000, &c. are in geometrical progression.

And their logarithms 0, 1, 2, 3, 4, &c. are in arithmetical progression.

Universally, if in any geometrical series,

a = the least term, r = the ratio,
 L = its logarithm, l = its logarithm;

Then the logarithm of ar is $L + l$, (Art. 1.)
 of ar^2 is $L + 2l$,
 of ar^3 is $L + 3l$, &c.

Here, the quantities $a, ar^2, ar^3, ar^4, \&c.$ are in geometrical progression. (Alg. 436.)

And their logarithms $L, L+l, L+2l, L+3l, \&c.$ are in arithmetical progression. (Alg. 423.)*

18. *Hyperbolic logarithms.* Although the system of logarithms which has now been explained, is far more convenient than any other, for the common purposes of calculation, and is the only one in general use, at the present time; yet it is not that which was first proposed, by the celebrated inventor of logarithms Lord Napier.† For particular reasons, he made the *radix* of his system 2.718281828459, instead of 10. This produces a change throughout the whole series, except the logarithm of 1, which, in every system, is 0. Napier's logarithms are also called *hyperbolic* logarithms, from certain relations which they have to the spaces between the asymptotes and the curve of an hyperbola; although these relations are not, in fact, peculiar to Napier's system. These logarithms have some particular uses, to which the common tables are not adapted.

THE LOGARITHMIC CURVE.

19. The relations of logarithms, and their corresponding numbers, may be represented by the abscissas and ordinates of a curve. Let the line AC (Fig. 1.) be taken for unity. Let AF be divided into portions, each equal to AC, by the points 1, 2, 3, &c. Let the line a represent the *radix* of a given system of logarithms, suppose it to be 1.3; and let $a^2, a^3, \&c.$ correspond, in length, with the different powers of a . Then the distances from A to 1, 2, 3, &c. will represent the *logarithms* of $a, a^2, a^3, \&c.$ (Art. 2.) The line CH is called the *logarithmic curve*, because its *abscissas* are proportioned to the logarithms of numbers represented by its *ordinates*. (Alg. 527.)

20. As the *abscissas* are the distances from AC, on the line

* See note C.

† It may not be correct to ascribe to Napier the original discovery of the *principle* of logarithms. But he was the first who constructed logarithmic tables, and adapted them to the methods of computation to which they are now so extensively applied. These tables were published in 1614. For a particular History of logarithms, see the Introduction to Hutton's Mathematical Tables.

AF, it is evident, that the abscissa of the point C is 0, which is the logarithm of $1=AC$. (Art. 2.) The distance from A to 1 is the logarithm of the ordinate a , which is the *radix* of the system. For Briggs' logarithms, this ought to be ten times AC. The distance from A to 2 is the logarithm of the ordinate a^2 ; from A to 3 is the logarithm of a^3 , &c.

21. The logarithms of numbers less than a unit are *negative*. (Art. 9.) These may be represented by portions of the line AN, on the *opposite side* of AC. (Alg. 507.) The ordinates a^{-1} , a^{-2} , a^{-3} , &c. are less than AC, which is taken for unity; and the abscissas, which are the distances from A to -1 , -2 , -3 , &c. are negative.

22. If the curve be continued ever so far, it will never meet the axis AN. For, as the ordinates are in geometrical progression decreasing, each is a certain portion of the preceding one. They will be diminished more and more, the farther they are carried, but can never be reduced absolutely to nothing. The axis AN is, therefore, an *asymptote* of the curve. (Alg. 545.) As the ordinate decreases, the abscissa increases; so that, when one becomes infinitely small, the other becomes infinitely great. This corresponds with what has been stated, (Art. 15.) that the logarithm of 0 is *infinite* and *negative*.

23. To find the *equation* of this curve,

Let a =the *radix* of the system,
 x =any one of the abscissas,
 y =the corresponding ordinate.

Then, by the nature of the curve, (Art. 19.) the *ordinate* to any point, is that power of a whose exponent is equal to the *abscissa* of the same point; that is (Alg. 528.)

$$y=a^x.*$$

*For other properties of the logarithmic curve, see *Fluxions*.

SECTION II.

DIRECTIONS FOR TAKING LOGARITHMS AND THEIR NUMBERS FROM THE TABLES.*

ART. 24. **T**HE purpose which logarithms are intended to answer, is to enable us to perform arithmetical operations with *greater expedition*, than by the common methods. Before any one can avail himself of this advantage, he must become so familiar with the tables, that he can readily find the logarithm of any number; and, on the other hand, the number to which any logarithm belongs.

In the common tables, the *indices* to the logarithms of the first 100 numbers, are inserted. But, for all other numbers, the *decimal* part only of the logarithm is given; while the index is left to be supplied, according to the principles in arts. 8 and 11.

25. *To find the logarithm of any number between 1 and 100;*

Look for the proposed number, on the left; and against it, in the next column, will be the logarithm, with its index. Thus

The log. of 18 is 1.25527. The log. of 73 is 1.86332.

26. *To find the logarithm of any number between 100 and 1000; or of any number consisting of not more than three significant figures, with ciphers annexed.*

In the smaller tables, the three first figures of each number, are generally placed in the left hand column; and the fourth figure is placed at the head of the other columns.

Any number, therefore, between 100 and 1000, may be found on the left hand; and directly opposite, in the next column, is the decimal part of its logarithm. To this the *index* must be prefixed, according to the rule in art. 8.

* The best English Tables are Hutton's in 8vo, and Taylor's in 4to. In these, the logarithms are carried to seven places of decimals, and proportional parts are placed in the margin. The smaller tables are numerous; and, when accurately printed, are sufficient for common calculations.

The log. of 458 is 2.66087, The log. of 935 is 2.97081,
 of 796 2.90091, of 386 2.58659.

If there are *ciphers* annexed to the significant figures, the logarithm may be found in a similar manner. For, by art. 14, the *decimal* part of the logarithm of any number is the same, as that of the number multiplied into 10, 100, &c. All the difference will be in the *index*; and this may be supplied by the same general rule.

The log. of 4580 is 3.66087, The log. of 326000 is 5.51322,
 of 79600 4.90091, of 8010000 6.90363.

27. *To find the logarithm of any number consisting of FOUR figures, either with, or without, ciphers annexed.*

Look for the three first figures, on the left hand, and for the fourth figure, at the head of one of the columns. The logarithm will be found, opposite the three first figures, and in the column which, at the head, is marked with the fourth figure.*

The log. of 6234 is 3.79477, The log. of 783400 is 5.89398,
 of 5231 3.71858, of 6281000 6.79803.

28. *To find the logarithm of a number containing MORE than FOUR significant figures.*

By turning to the tables, it will be seen, that if the *differences* between several numbers be small, in comparison with the numbers themselves; the differences of the *logarithms* will be nearly proportioned to the differences of the *numbers*. Thus

The log. of 1000 is 3.00000,	
of 1001 3.00043,	Here the differences in
of 1002 3.00087,	the numbers are 1, 2, 3, 4, &c.
of 1003 3.00130,	and the corresponding dif-
of 1004 3.00173, &c.	ferences in the logarithms
	are 43, 87, 130, 173, &c.

Now 43 is nearly half of 87, one third of 130, one fourth of 173, &c.

Upon this principle, we may find the logarithm of a number which is between two other numbers whose logarithms are given by the tables. Thus the logarithm of 21716 is not to be found, in those tables which give the numbers to four places of figures only.

* In Taylor's, Hutton's and other tables, *four* figures are placed in the left hand column, and the *fifth* at the top of the page.

But by the table, the log. of 21720 is 4.33686
and the log. of 21710 is 4.33666

The difference of the two numbers is 10; and that of the logarithms 20.

Also, the difference between 21710, and the proposed number 21716 is 6.

If, then, a difference of 10 in the numbers
make a difference of 20 in the logarithms:

A difference of 6 in the numbers, will
make a difference of 12 in the logarithms.

That is, $10 : 20 :: 6 : 12$.

If, therefore, 12 be added to 4.33666, the log. of 21710;
The sum will be 4.33678, the log. of 21716.

We have, then, this

RULE.

To find the logarithm of a number consisting of more than four figures;

Take out the logarithm of two numbers, one greater, and the other less, than the number proposed: Find the difference of the two numbers, and the difference of their logarithms: Take also the difference between the least of the two numbers, and the proposed number. Then say,

As the difference of the two numbers,

To the difference of their logarithms;

So is the difference between the least of the two numbers, and the proposed number,

To the proportional part to be added
to the least of the two logarithms.

It will generally be expedient to make the *four first figures*, in the least of the two numbers, the same as in the proposed number, substituting ciphers, for the remaining figures; and to make the greater number the same as the less, with the addition of a unit to the last significant figure. Thus

For 36843, take 36840, and 36850,

For 792674, 792600, 792700,

For 6537825, 6537000, 6538000, &c.

The first term of the proportion will then be 10, or 100, or 1000, &c.

Ex. 1. Required the logarithm of 362572.

The logarithm of 362600 is 5.55943
of 362500 5.55931

The differences are 100, and 12.

Then $100 : 12 :: 72 : 8.64$, or 9 nearly.

And the log. $5.55931 + 9 = 5.55940$, the log. required.

Ex. 2. The log. of 78264 is 4.89356

3. The log. of 143542 is 5.15698

4. The log. of 1129535 is 6.05290.

By a little practice, such a facility, in abridging these calculations, may be acquired, that the logarithms may be taken out, in a very short time. When great accuracy is not required, it will be easy to make an allowance sufficiently near, without formally stating a proportion. In the larger tables, the proportional parts which are to be added to the logarithms, are already prepared, and placed in the margin.

29. *To find the logarithm of a DECIMAL FRACTION.*

The logarithm of a decimal is the same as that of a whole number, excepting the *index*. (Art. 14.) To find then the logarithm of a decimal, take out that of a whole number consisting of the same figures; *observing to make the negative index equal to the distance of the first significant figure of the fraction from the place of units*. (Art. 11.)

The log. of 0.07643, is $\overline{2}.88326$, or 8.88326, (Art. 12.)
of 0.00259, $\overline{3}.41330$, or 7.41330,
of 0.0006278, $\overline{4}.79782$, or 6.79782.

30. *To find the logarithm of a MIXED decimal number.*

Find the logarithm, in the same manner as if *all* the figures were integers; and then prefix the index which belongs to the *integral* part, according to art. 8.

The logarithm of 26.34 is 1.42062.

The index here is 1, because 1 is the index of the logarithm of every number greater than 10, and less than 100. (Art. 7.)

The log. of 2.36 is 0.37291, The log. of 364.2 is 2.56134,
of 27.8 1.44404, of 69.42 1.84146.

31. *To find the logarithm of a VULGAR FRACTION.*

From the nature of a vulgar fraction, the numerator may be considered as a *dividend*, and the denominator as a *divisor*; in other words, the value of the fraction is equal to the quotient, of the numerator divided by the denominator. (Alg. 135.) But in logarithms, division is performed by *subtraction*; that is, the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of those numbers. (Art. 1.) To find then the logarithm of a vulgar fraction, *subtract the logarithm of the denominator from that of the numerator*. The difference will be the logarithm of the fraction. Or the logarithm may be found, by first reducing the vulgar fraction to a *decimal*. If the numerator is less than the denominator, the index of the logarithm must be *negative*, because the value of the fraction is less than a unit. (Art. 9.)

Required the logarithm of $\frac{3}{8}\frac{4}{7}$.

The log. of the numerator is 1.53146
of the denominator 1.93952

of the fraction	<u>1.59196</u> , or 9.59196.
The logarithm of $\frac{3}{7}\frac{6}{8}\frac{2}{3}\frac{4}{4}$ is	<u>2.66362</u> , or 8.66362.
of $\frac{7}{8}\frac{3}{3}\frac{9}{9}$	<u>3.04376</u> , or 7.04376.

32. If the logarithm of a *mixed number* is required, reduce it to an improper fraction, and then proceed as before.

The logarithm of $3\frac{1}{5} = \frac{16}{5}$ is 0.57724.

33. *To find the NATURAL NUMBER belonging to any logarithm.*

In computing by logarithms, it is necessary, in the first place, to take from the tables the logarithms of the numbers which enter into the calculation; and, on the other hand, at the close of the operation, to find the number belonging to the logarithm obtained in the result. This is evidently done, by *reversing* the methods in the preceding articles.

Where great accuracy is not required, look in the tables for the logarithm which is *nearest* to the given one; and directly opposite, on the left hand, will be found the *three first fig-*

ures, and at the top, over the logarithm, the *fourth* figure, of the number required. This number, by pointing off decimals, or by adding ciphers, if necessary, must be made to correspond with the *index* of the given logarithm, according to arts. 8 and 11.

The natural number belonging

to 3.86493 is 7327, to 1.62572 is 42.24,
to 2.90147 796.9, to 2.89115 0.07783.

0

In the last example, the index requires that the first significant figure should be in the *second* place from units, and therefore a cipher must be prefixed. In other instances, it is necessary to annex ciphers on the *right*, so as to make the number of figures exceed the index by 1.

The natural number belonging

to 6.71567 is 5196000, to 3.65677 is 0.004537,
to 4.67062 46840, to 4.59802 0.0003963.

34. When great accuracy is required, and the given logarithm is not exactly, or very nearly, found in the tables, it will be necessary to reverse the rule in art. 28.

Take from the tables two logarithms, one the next greater, the other the next less than the given logarithm. Find the difference of the two logarithms, and the difference of their natural numbers; also the difference between the least of the two logarithms, and the given logarithm. Then say,

As the difference of the two logarithms,
To the difference of their numbers;
So is the difference between the given
logarithm and the least of the other two,
To the proportional part to be added to
the least of the two numbers.

Required the number belonging to the logarithm 2.67325.

Next great.log. 2.67330. Its numb. 471.3. Given log. 2.67325.

Next less 2.67321. Its numb. 471.2. Next less 2.67321.

Differences	<u>9</u>	<u>0.1</u>	<u>4</u>
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Then $9:0.1::4:0.044$, which is to be added
to the number 471.2

The number required is 471.244.

The natural number belonging
to 4.37627 is 23783.45, to 1.73698 is 54.57357,
to 3.69479 4952.08, to 1.09214 is 0.123635.

35. *Correction of the tables.* The tables of logarithms have been so carefully and so repeatedly calculated, by the ablest computers, that there is no room left to question their general correctness. They are not, however, exempt from the common imperfections of the press. But an error of this kind is easily corrected, by comparing the logarithm with any two others to whose *sum* or *difference* it ought to be equal. (Art. 1.)

Thus $48 = 24 \times 2 = 16 \times 3 = 12 \times 4 = 8 \times 6$. Therefore, the logarithm of 48 is equal to the *sum* of the logarithms of 24 and 2, of 16 and 3, &c.

And $3 = \frac{6}{2} = \frac{12}{4} = \frac{18}{6} = \frac{24}{8} = \frac{36}{12}$, &c. Therefore, the logarithm of 3 is equal to the *difference* of the logarithms of 6 and 2, of 12 and 4, &c.

SECTION III.

METHODS OF CALCULATING BY LOGARITHMS.

ART. 36. **T**HE arithmetical operations for which logarithms were originally contrived, and on which their great utility depends, are chiefly multiplication, division, involution, evolution, and finding the term required in single and compound proportion. The principle on which all these calculations are conducted, is this;

If the logarithm of two numbers be added, the SUM will be the logarithm of the PRODUCT of the numbers; and

If the logarithm of one number be subtracted from that of another, the DIFFERENCE will be the logarithm of the QUOTIENT of one of the numbers divided by the other.

In proof of this, we have only to call to mind, that logarithms are the EXPONENTS of a series of powers and roots. (Arts. 2, 5.) And it has been shown, that powers and roots are multiplied, by adding their exponents; and divided, by subtracting their exponents. (Alg. 233, 237, 280, 286.)

MULTIPLICATION BY LOGARITHMS.

37. Add the logarithms of the FACTORS: the SUM will be the logarithm of the PRODUCT.

In making the addition, 1 is to be carried, for every 10, from the decimal part of the logarithm, to the index. (Art. 7.)

Numbers.	Logarithms.	Numbers.	Logarithms.
Mult. 36.2 (Art. 30.)	1.55871.	Mult. 640	2.80618
Into 7.84	0.89432.	Into 2.316	0.36474
Prod. 283.8	2.45303	Prod. 1482	3.17092

The logarithms of the two factors are taken from the ta-

MULTIPLICATION BY

bles. The product is obtained, by finding, in the tables, the natural number belonging to the sum. (Art. 33.)

Mult. 89.24	1.95056	Mult. 134.	2.12710
Into 3.687	0.56667	Into 25.6	1.40824
Prod. 329.	2.51723	Prod 3430.	3.53534

38. When any or all of the indices of the logarithms are *negative*, they are to be added according to the rules for the addition of positive and negative quantities in algebra. But it must be kept in mind, that the *decimal* part of the logarithm is *positive*. (Art. 10.) Therefore, that which is carried from the decimal part to the index, must be considered positive also:

Mult. 62.84	1.79824	Mult. 0.0294	2.46835
Into 0.682	1.83376	Into 0.8372	1.92283
Prod. 42.86	1.63202	Prod. 0.0246	2.39118

In each of these examples, +1 is to be carried from the decimal part of the logarithm. This added to -1, the lower index, makes it 0; so that there is nothing to be added to the upper index.

If any perplexity is occasioned, by the addition of positive and negative quantities, it may be avoided, by borrowing 10 to the index. (Art. 12.)

Mult. 62.84	1.79824	Mult. 0.0294	8.46835
Into 0.682	9.83378	Into 0.8372	9.92283
Prod. 42.86	1.63202	Prod. 0.0246	8.39118

Here 10 is added to the negative indices, and afterwards rejected from the index of the sum of the logarithms.

Multiply 26.83	1.42862	1.42862
Into 0.00069	4.83885	or 6.83885
Product 0.0185	2.26747	8.26747

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Here +1 carried to -4 makes it -3, which added to the upper index +1, gives -2 for the index of the sum.

Multiply	.00845	3.92686	or	7.92686
Into	1068.	3.02857		3.02857
Product	<u>9.0246</u>	<u>0.95543</u>		<u>0.95543</u>

The product of 0.0362 into 25.38 is 0.9188
of 0.00467 into 348.1 is 1.626*
of 0.0861 into 0.00843 is 0.0007258 +

39. Any number of factors may be multiplied together, by adding their logarithms. If there are several positive, and several negative indices, these are to be reduced to one, as in algebra, by taking the difference between the sum of those which are negative, and the sum of those which are positive, increased by what is carried from the decimal part of the logarithms. (Alg. 78.)

Multiply	6832	3.83455		3.83455
Into	0.00863	3.93601	or	7.93601
And	0.651	1.81356		0.81358
And	0.0231	2.36361	or	8.36361
And	62.87	1.79844		1.79844
Prod.	<u>55.74</u>	<u>1.74619</u>		<u>1.74619</u>

Here the sum of the positive indices is 4
To which, adding what is carried from the dec. part 3

And subtracting the sum of the negative indices

The remainder is 7 the index of the sum of the logarithms.

Or, if 10 be added to each of the two negative indices, 30 must be taken from the index of the sum, and the answer will be the same, as by the other method.

* This sh^d be 1.626; in

+ With 5 places .000726*

Ex. 2. The prod. of $36.4 \times 7.82 \times 68.91 \times 0.3846$ is 7544.

3. The prod. of $0.00629 \times 2.647 \times 0.082 \times 278.8 \times 0.00063$ is 0.000567.

40. *Negative* quantities are multiplied, by means of logarithms, in the same manner as those which are positive. (Art. 16.) But, after the operation is ended, the proper sign must be applied to the natural number expressing the product, according to the rules for the multiplication of positive and negative quantities in algebra. The negative index of a *logarithm*, must not be confounded with the sign which denotes that the *natural number* is negative. That which the index of the logarithm is intended to show, is not whether the natural number is *positive or negative*, but whether it is *greater or less than a unit*. (Art. 16.)

Mult. +36.42	1.56134	Mult. -2.681	0.42830
Into -67.31	1.82808	Into +37.24	1.57101
Prod. -2451	3.38942	Prod. -99.84	1.99931

In these examples, the logarithms are taken from the tables, and added, in the same manner, as if both factors were positive. But after the product is found, the negative sign is prefixed to it, because + is multiplied into -. (Alg. 105.)

Mult. 0.263	1.41996	Mult. 0.065	2.81291
Into 0.00894	3.95134	Into 0.693	1.84073
Prod. 0.002351	3.37130	Prod. 0.04504	2.65364

Here, the indices of the logarithms are negative, but the product is positive, because the factors are both positive.

Mult. -62.59	1.79650	Mult. -68.3	1.83142
Into -0.00863	3.93601	Into -0.0096	3.98227
Prod. +0.5402	1.73251	Prod. +0.6557	1.81669

DIVISION BY LOGARITHMS.

41. From the logarithm of the DIVIDEND, SUBTRACT the logarithm of the DIVISOR; the DIFFERENCE will be the logarithm of the QUOTIENT. (Art. 36.)

Numbers.	Logarithms.	Numbers.	Logarithms.
Divide 6238	3.79505	Divide 896.3	2.95245
By 2982	3.47451	By 9.847	0.99330
Quot. 2.092	0.32054	Quot. 91.02	1.95915

42. The decimal part of the logarithm may be subtracted as in common arithmetic. But for the *indices*, when either of them is negative, or the lower one is greater than the upper one, it will be necessary to make use of the general rule for subtraction in algebra; that is, to change the signs of the subtrahend, and then proceed as in addition. (Alg. 82.) When 1 is carried from the decimal part, this is to be considered affirmative, and applied to the index, before the sign is changed.

Divide 0.8697	1.93937	or	9.93937
By 98.65	1.99410		1.99410
Quot. 0.006816	3.94527		7.94527

In this example, the upper logarithm being less than the lower one, it is necessary to borrow 10, as in other cases of subtraction; and therefore to carry 1 to the lower index, which then becomes +2. This changed to -2, and added to -1 above it, makes the index of the difference of the logarithms -3.

Divide 29.76	1.47363	1.47363
By 6254	3.79616	3.79616
Quot. 0.00476	3.67747	or 7.67747

Here, 1 carried to the lower index, makes it +4. This

changed to -4 , and added to 1 above it, gives -3 for the index of the difference of the logarithms.

Divide 6.832	0.83455	Divide 0.00634	3.80209
By .0362	2.55871	By 62.18	1.79365
Quot. 188.73	2.27584	Quot. 0.000102	4.00844

The quotient of 0.0965 divided by 0.007241, is 13.6.

The quotient of 0.0621 divided by 3.68, is 0.01687.

43. To divide *negative* quantities, proceed in the same manner as if they were positive, (Art. 40.) and prefix to the quotient, the sign which is required by the rules for division in algebra.

Divide +3642	3.56134	Divide -0.657	1.81757
By -23.68	1.37436	By +0.0793	2.89927
Quot. -153.8	2.18696	Quot. -8.285	0.91830

In these examples, the sign of the divisor being different from that of the dividend, the sign of the quotient must be negative. (Alg. 123.)

Divide -0.364	1.56110	Divide -62.5	1.83569
By -2.56	0.40824	By +0.094	2.97313
Quot. +0.1422	1.15286	Quot. -728.7	2.86256

INVOLUTION BY LOGARITHMS.

44. Involving a quantity is multiplying it into itself. By means of logarithms, multiplication is performed by addition. If, then, the logarithm of any quantity be added to itself, the logarithm of a *power* of that quantity will be obtained. But adding a logarithm, or any other quantity, to itself, is *multiplication*. The involution of quantities, by means of logarithms, is therefore performed, by multiplying the logarithms.

Thus the logarithm
of 100

is 2
of 100×100 , that is, of 100^2 is $2+2 = 2 \times 2$.
of $100 \times 100 \times 100$, 100^3 is $2+2+2 = 2 \times 3$.
of $100 \times 100 \times 100 \times 100$, 100^4 is $2+2+2+2 = 2 \times 4$.

On the same principle, the logarithm of 100^n is $2 \times n$.
And the logarithm of x^n , is $(\log. x) \times n$. Hence,

45. To involve a quantity by logarithms. **MULTIPLY the logarithm of the quantity, by the INDEX of the power required.**
The reason of the rule is also evident, from the consideration, that logarithms are the exponents of powers and roots, and a power or root is involved, by *multiplying* its index into the index of the power required. (Alg. 220, 288.)

Ex. 1. What is the cube of 6.29# ?

Root 6.296, its log. 0.79906
Index of the power 3

Power 249.6 5.39718

2. Required the 4th power of 21.32

Root 21.32 log. 1.32879
Index 4

Power 206614 5.31516

3. Required the 6th power of 1.689

Root 1.689 log. 0.22763
Index 6

Power 23.215 1.36578

4. Required the 144th power of 1.003

Root 1.003 log. 0.00130
Index 144

Power 1.539 0.18720

46. It must be observed, as in the case of multiplication,

(Art. 36.) that what is carried from the *decimal* part of the logarithm is *positive*, whether the index itself is positive or negative. Or, if 10 be added to a negative index, to render it positive, (Art. 12.) this will be multiplied, as well as the other figures, so that the logarithm of the square, will be 20 too great; of the cube, 30 too great, &c.

Ex. 1. Required the cube of 0.0649

Root	0.0649	log.	<u>2.81224</u>	or	<u>8.81224</u>
		Index	3		3
Power	0.0002733		<u>4.43672</u>		<u>6.43672</u>

The index -2 , multiplied by 3, becomes -6 . But $+2$ carried from the decimal part, reduces it to -4 .

Or, if 10 be added to the index, so as to make it $+6$; when this is multiplied by 3, and the product increased by 2 carried from the decimal part, it becomes 26, which is 30 too great. Leaving off the first figure, it still remains 10 too great, $+6$ instead of -4 .

2. Required the 4th power of 0.1234

Root	0.1234	log.	<u>1.09132</u>	or	<u>9.09132</u>
		Index	4		4
Power	0.0002319		<u>4.36528</u>		<u>6.36528</u>

3. Required the 6th power of 0.9977

Root	0.9977	log.	<u>1.99900</u>	or	<u>9.99900</u>
		Index	6		6
Power	0.9863		<u>1.99400</u>		<u>9.99400</u>

4. Required the cube of 0.08762.

Root	0.08762	log.	<u>2.94260</u>	or	<u>8.94260</u>
		Index	3		3
Power	0.0006727		<u>4.82780</u>		<u>6.82780</u>

5. The 7th power of 0.9061 is 0.5015.

6. The 5th power of 0.9344 is 0.7123.

EVOLUTION BY LOGARITHMS.

47. Evolution is the opposite of involution. Therefore, as quantities are involved, by the *multiplication* of logarithms, roots are extracted by the *division* of logarithms ; that is,

To extract the root of a quantity by logarithms, *DIVIDE the logarithm of the quantity, by the number expressing the root required.*

The reason of the rule is evident also, from the fact, that logarithms are the exponents of powers and roots, and evolution is performed, by dividing the exponent, by the number expressing the root required. (Alg. 257.)

1. Required the square root of 648.3.

	Numbers.	Logarithms.
Power	648.3	2)2.81178
Root	25.46	1.40589

2. Required the cube root of 897.1.

Power	897.1	3)2.95284
Root	9.645	0.98428

In the first of these examples, the logarithm of the given number is divided by 2 ; in the other, by 3.

3. Required the 10th root of 6948.

Power	6948	10)3.84166
Root	2.422	0.38418

4. Required the 100th root of 983.

Power	983	100)2.99255
Root	1.071	0.02992

The division is performed here, as in other cases of decimals, by removing the decimal point to the left.

5. What is the ten thousandth root of 49680000 ?

Power	49680000	10000)7.69618
Root	1.00179	0.00077

We have, here, an example of the great rapidity with which arithmetical operations are performed by logarithms.

E

48. If the index of the logarithm is *negative*, and is *not divisible* by the given divisor, without a remainder, a difficulty will occur, unless the index be altered.

Suppose the cube root of 0.0000892 is required. The logarithm of this is $\overline{5.95036}$. If we divide the index by 3, the quotient will be -1 ; with -2 remainder. This remainder, if it were positive, might, as in other cases of division, be prefixed to the next figure. But the remainder is *negative*, while the decimal part of the logarithm is *positive*; so that, when the former is prefixed to the latter, it will make neither $+2.9$ nor -2.9 , but $-2+.9$. This embarrassing intermixture of positives and negatives may be avoided, by adding to the index another negative number, to make it exactly divisible by the divisor. Thus, if to the index -5 there be added -1 , the sum -6 will be divisible by 3. But this addition of a negative number must be *compensated*, by the addition of an equal positive number, which may be prefixed to the decimal part of the logarithm. The division may then be continued, without difficulty, through the whole.

Thus, if the logarithm $\overline{5.95036}$ be altered to $\overline{6+1.95036}$ it may be divided by 3, and the quotient will be $\overline{2.65012}$. We have then this rule,

49. Add to the index, if necessary, such a negative number as will make it exactly divisible by the divisor, and prefix an equal positive number to the decimal part of the logarithm.

1. Required the 5th root of 0.009642

Power 0.009642	log. $\overline{3.98417}$
	or $\overline{5} + \overline{2.98417}$
Root 0.3952	$\overline{1.59683}$

2. Required the 7th root of 0.0004935

Power 0.0004935	log. $\overline{4.69329}$
	or $\overline{7} + \overline{3.69329}$
Root 0.337	$\overline{1.52761}$

50. If, for the sake of performing the division conveniently, the negative index be rendered *positive*, it will be expedient to borrow as many tens, as there are units in the number denoting the root.

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What is the fourth root of 0.03698?

Power 0.03698	4) $\overline{2.56797}$	or 4) 38.56797
Root 0.4385	1.64199	9.64199

Here the index, by borrowing, is made 40 ~~too~~ great, that is, +38 instead of -2. When, therefore, it is divided by 4, it is still 10 too great, +9 instead of -1.

What is the 5th root of 0.008926?

Power 0.008926	5) $\overline{3.95066}$	or 5) 47.95066
Root 0.38916	1.59013	9.59013

51. A *power of a root* may be found by first *multiplying* the logarithm of the given quantity into the index of the power, (Art. 45.) and then *dividing* the product by the number expressing the root. (Art. 47.)

1. What is the value of $(53)^{\frac{6}{7}}$, that is, the 6th power of the 7th root of 53?

Given number 53		log. 1.72428
Multiplying by		6
Dividing by		7) 10.34568
Power required 30.06		1.47795

2. What is the 8th power of the 9th root of 654?

PROPORTION BY LOGARITHMS.

52. In a proportion, when three terms are given, the fourth is found, in common arithmetic, by multiplying together the second and third, and dividing by the first. But, when logarithms are used, *addition* takes the place of multiplication, and *subtraction*, of division.

To find then, by logarithms, the fourth term in a proportion, *ADD the logarithms of the SECOND and THIRD terms, and from the sum SUBTRACT the logarithm of the FIRST term.* The remainder will be the logarithm of the term required.

Ex. 1. Find a fourth proportional to 7964, 378, and 27960.

	Numbers.	Logarithms.
Second term	378	2.57749
Third term	27960	4.44654
First term	7964	7.02403
		3.90113
Fourth term	1327	3.12290

2. Find a 4th proportional to 768, 381 and 9760.

Second term	381	2.58092
Third term	9780	3.99034
First term	768	6.57126
		2.88536
Fourth term	4852	3.68590

ARITHMETICAL COMPLEMENT.

53. When one number is to be subtracted from another, it is often convenient, first to subtract it from 10, then to *add the difference* to the other number, and afterwards to reject the 10.

Thus, instead of $a - b$, we may put $10 - b + a - 10$.

In the first of these expressions, b is subtracted from a . In the other, b is subtracted from 10, the difference is added to a , and 10 is afterwards taken from the sum. The two expressions are equivalent, because they consist of the same terms, with the addition, in one of them, of $10 - 10 = 0$. The alteration is, in fact, nothing more than borrowing 10, for the sake of convenience, and then rejecting it in the result.

Instead of 10, we may borrow, as occasion requires, 100, 1000, &c.

Thus $a - b = 100 - b + a - 100 = 1000 - b + a - 1000$, &c.

54. The DIFFERENCE between a given number and 10, or 100, or 1000, &c. is called the ARITHMETICAL COMPLEMENT of that number.

The arithmetical complement of a number consisting of one integral figure, either with or without decimals, is found, by subtracting the number from 10. If there are two inte-

gral figures, they are subtracted from 100; if *three*, from 1000, &c.

Thus the arithmetical compl't of 3.46 is $10 - 3.46 = 6.54$
of 34.6 is $100 - 34.6 = 65.4$
of 346. is $1000 - 346. = 654. \&c.$

According to the rule for subtraction in arithmetic, any number is subtracted from 10, 100, 1000, &c. by beginning on the right hand, and taking each figure from 10, after *increasing* all except the first, by *carrying* 1.

Thus, if from	10.00000
We subtract	7.63125

The difference, or arith'l comp't is 2.36875, which is obtained, by taking 5 from 10, 3 from 10, 2 from 10, 4 from 10, 7 from 10, and 8 from 10. But, instead of taking each figure, *increased* by 1, from 10; we may take it *without being increased*, from 9.

Thus 2 from 9 is the same as 3 from 10,
3 from 9, the same as 4 from 10, &c. Hence,

55. *To obtain the ARITHMETICAL COMPLEMENT of a number, subtract the right hand significant figure from 10, and each of the other figures from 9.* If, however, there are *iphers* on the right hand of all the significant figures, they are to be set down without alteration.

In taking the arithmetical complement of a logarithm, if the *index* is *negative*, it must be *added* to 9; for adding a negative quantity is the *same* as subtracting a positive one. (Alg. 81.) The difference between -3 and $+9$, is not 6, but 12.

The arithmetical complement

of 6.24897 is 3.75103	of $\overline{2.70649}$ is 11.29351
of 2.98643 7.01357	of 3.64200 6.35800
of 0.62430 9.37570	of 9.35001 0.64999.

56. The principal use of the arithmetical complement, is in working proportions by logarithms; where some of the terms are to be *added*, and one or more to be *subtracted*. In the Rule of Three or simple proportion, two terms are to be added, and from the sum, the first term is to be subtracted.

But if, instead of the logarithm of the first term, we substitute its arithmetical complement, this may be *added* to the sum of the other two, or more simply, all three may be added together, by one operation. After the index is diminished by 10, the result will be the same as by the common method. For subtracting a number is the same, as adding its arithmetical complement, and then rejecting 10, 100, or 1000, from the sum. (Art. 53.)

In the following proportion, the calculation is made in both ways.

If the profit on 2625 dollars employed in trade, is 831 dollars; what is the profit on 536 dollars?

By the common method.		By the Arith ^l Complement.	
2d term 831	2.91960	2d term	2.91960
3d term 536	2.72916	3d term	2.72916
	5.64876		5.64876
1st term 2625	3.41913	1st term a. c.	6.58087
4th term 169.68	2.22963	4th term	2.22963

The second method here, after rejecting 10, gives the same result as the first. But it is unnecessary, first to add two of the terms, and then the arithmetical complement of the other. The three may be added at once; and it will generally be expedient, to place the terms in the same order, in which they are arranged in the statement of the proportion.

Thus, As 2625\$ stock,	a. c.	6.58087
Is to 831 profit		2.91960
So is 536 stock		2.72916
To 169.68 profit		2.22963

2. As 6273 a. c.	6.20252	3. As 253 a. c.	7.59688
Is to 769.4	2.88615	Is to 672.5	2.82769
So is 37.61	1.57530	So is 497	2.69636
To 4.613	0.66397	To 1321.1	3.12063

COMPOUND PROPORTION.

31

4. As 46.34 a. c. 8.33404	5. As 9.85 a. c. 9.00656
Is to 892.1 2.95041	Is to 643 2.80821
So is 7.636 0.89298	So is 76.3 1.86252
To 147 2.16743	To 4981 3.69729

COMPOUND PROPORTION.

57. In compound, as in single proportion, the term required may be found by logarithms, if we substitute addition for multiplication, and subtraction for division.

Ex. 1. If the interest of \$365, for 3 years and 9 months, be \$82.13; what will be the interest of \$8940, for 2 years and 6 months?

In common arithmetic, the statement of the question is made in this manner,

$$\begin{array}{l} 365 \text{ dollars} \\ 3.75 \text{ years} \end{array} \left. \vphantom{\begin{array}{l} 365 \text{ dollars} \\ 3.75 \text{ years} \end{array}} \right\} : 82.13 \text{ dollars} :: \left\{ \begin{array}{l} 8940 \text{ dollars} \\ 2.5 \text{ years} \end{array} \right\} :$$

And the method of calculation is, to *divide* the *product* of the third, fourth, and fifth terms, by the *product* of the two first.* This, if logarithms are used, will be to *subtract* the *sum* of the logarithms of the two first terms, from the *sum* of the logarithms of the other three.

Two first terms	$\left\{ \begin{array}{l} 365 \text{ log.} \\ 3.75 \end{array} \right.$	$\begin{array}{r} 2.56229 \\ 0.57403 \\ \hline \end{array}$
Sum of the logarithms		$\begin{array}{r} 3.13632 \\ \hline \end{array}$
Third term	82.13	1.91450
Fourth and fifth terms	$\left\{ \begin{array}{l} 8940 \\ 2.5 \end{array} \right.$	$\begin{array}{r} 3.95134 \\ 0.39794 \\ \hline \end{array}$
Sum of the logs. of the 3d, 4th, and 5th		6.26378
Do.	1st and 2d	$\begin{array}{r} 3.13632 \\ \hline \end{array}$
Term required	1341	$\begin{array}{r} 3.12746 \\ \hline \end{array}$

* See Webber's Arithmetic.

58. The calculation will be more simple, if, instead of *subtracting* the logarithms of the two first terms, we *add* their *arithmetical complements*. But it must be observed, that *each* arithmetical complement increases the index of the logarithm by 10. If the arithmetical complement be introduced into *two* of the terms, the index of the sum of the logarithms will be 20 too great; if it be in *three* terms, the index will be 30 too great, &c.

Two first terms	{ 365	a. c. 7.43771
	{ 3.75	a. c. 9.42597
Third term	82.13	1.91450
Fourth and fifth terms	{ 8940	3.95134
	{ 2.5	0.39794
Term required	1341	23.12746

The result is the same as before, except that the index of the logarithm is 20 too great.

Ex. 2. If the wages of 53 men for 42 days be 2200 dollars; what will be the wages of 87 men for 34 days?

53 men } : 2200 :: { 87 men }	
42 days }	{ 34 days } :
Two first terms { 53	a. c. 8.27572
{ 42	a. c. 8.37675
Third term 2200	3.34242
Fourth and fifth terms { 87	1.93952
{ 34	1.53148
Term required 2923.5	3.46589

59. In the same manner, if the product of *any number* of quantities, is to be divided, by the product of several others; we may add together the logarithms of the quantities to be divided, and the arithmetical complements of the logarithms of the divisors.

Ex. If 29.67×346.2 be divided by $69.24 \times 7.862 \times 497$; what will be the quotient?

Numbers to be divided	{ 29.67	1.47232
	{ 346.2	2.53933
Divisors	{ 69.24	a. c. 8.15964
	{ 7.862	a. c. 9.10447
	{ 497	a. c. 7.30364
Quotient	0.03797	<u>8.57940</u>

In this way, the calculations in *Conjoined Proportion* may be expeditiously performed.

COMPOUND INTEREST.

60. In calculating compound interest, the amount for the first year, is made the principal for the second year; the amount for the second year, the principal for the third year, &c. Now the amount at the end of each year, must be proportioned to the principal at the beginning of the year. If the principal for the first year be 1 dollar, and if the amount of 1 dollar for 1 year $=a$; then, (Alg. 377.)

$$1:a:: \begin{cases} a : a^2 = \text{the am't for the 2d y'r, or the prin. for the 3d;} \\ a^2 : a^3 = \text{the am't for the 3d y'r, or the prin. for the 4th;} \\ a^3 : a^4 = \text{the am't for the 4th y'r, or the prin. for the 5th,} \end{cases} \quad \text{[&c.]}$$

That is, the amount of 1 dollar for any number of years is obtained, by finding the amount for 1 year, and involving this to a power whose index is equal to the number of years. And the amount of any other principal, for the given time, is found, by multiplying the amount of 1 dollar, into the number of dollars, or the fractional part of a dollar.

If logarithms are used, the *multiplication* required here may be performed by *addition*; and the *involution*, by *multiplication*. (Art. 45.) Hence,

61. To calculate Compound Interest, Find the amount of 1 dollar for 1 year; multiply its logarithm by the number of years; and to the product, add the logarithm of the principal. The sum will be the logarithm of the amount for the given time. From the amount subtract the principal, and the remainder will be the interest.

If the interest becomes due *half-yearly* or *quarterly*; find the amount of one dollar, for the half-year or quarter, and

multiply the logarithm, by the number of half-years or quarters in the given time.

Ex. 1. What is the amount of 20 dollars, at 6 per cent compound interest, for 100 years?

Amount of 1 dollar for 1 year		1.06	log. 0.0253059
Multiplying by			100
Given principal	20		2.53059
Amount required	\$6786		1.30103
			3.83162

2. What is the amount of 1 cent, at 6 per cent compound interest, in 500 years?

Amount of 1 dollar for 1 year		1.06	log. 0.0253059
Multiplying by			500
Given principal	0.01		12.65295
Amount	\$44,973,000,000		-2.00000
			10.65295

More exact answers may be obtained, by using logarithms of a greater number of decimal places.

3. What is the amount of 1000 dollars, at 6 per cent compound interest, for 10 years? Ans. 1790.80.

EXPONENTIAL EQUATIONS.

62. An EXPONENTIAL equation is one in which the letter expressing the unknown quantity is an *exponent*.

Thus $a^x = b$, and $x^a = bc$, are exponential equations. These are most easily solved by logarithms. As the two members of an equation are equal, their logarithms must also be equal. If the logarithm of each side be taken, the equation may then be reduced, by the rules given in algebra.

Ex. What is the value of x , in the equation $3^x = 243$?

Taking the logarithms of both sides $\log. (3^x) = \log. 243$
 But the logarithm of a *power* is equal to the logarithm of
 the root, multiplied into the index of the power. (Art. 45.)

Therefore $(\log. 3) \times x = \log. 243$; and dividing by $\log. 3$,

$$x = \frac{\log. 243}{\log. 3} = \frac{2.38561}{0.47712} = 5. \text{ So that } 3^5 = 243.$$

63. The preceding is an exponential equation of the simplest form. Other cases, after the logarithm of each side is taken, may be solved by *Trial and Error*, in the same manner as affected equations. (Alg. 503.) For this purpose, make two suppositions of the value of the unknown quantity, and find their errors; then say,

As the difference of the errors, to the difference of the assumed numbers;

So is the least error, to the correction required in the corresponding assumed number.

Ex. 1. Find the value of x in the equation $x^x = 256$.

Taking the logarithms of both sides $(\log. x) \times x = \log. 256$

Let x be supposed equal to 3.5, or 3.6.

By the first supposition.
 $x = 3.5$, and $\log. x = 0.54407$

Multiplying by 3.5

$$\begin{array}{r} (\log. x) \times x = 1.90424 \\ \log. 256 = 2.40824 \\ \hline \end{array}$$

Error -0.50400

Difference of the errors

By the second supposition.

$x = 3.6$, and $\log. x = 0.55630$

Multiplying by 3.6

$$\begin{array}{r} (\log. x) \times x = 2.00268 \\ \log. 256 = 2.40824 \\ \hline \end{array}$$

Error -0.40556

0.09844

Then $0.09844 : 0.1 :: 0.40556 : 0.4119$, the correction. This added to 3.6, the second assumed number, makes the value of $x = 4.0119$.

To correct this farther, suppose $x = 4.011$, or 4.012.

By the first supposition,
 $x = 4.011$, and $\log. x = 0.60325$

Multiplying by 4.011

$$\begin{array}{r} (\log. x) \times x = 2.41963 \\ \log. 256 = 2.40824 \\ \hline \end{array}$$

Error +0.01139

Difference of the errors

By the second supposition,
 $x = 4.012$, and $\log. x = 0.60336$

Multiplying by 4.012

$$\begin{array}{r} (\log. x) \times x = 2.42068 \\ \log. 256 = 2.40824 \\ \hline \end{array}$$

Error +0.01244

0.00105

Then $0.00105 : 0.001 :: 0.01139 : 0.011$ very nearly. $\frac{1}{2}$

Subtracting this correction from the first assumed number 4.011, we have the value of $x=4$, which satisfies the conditions of the proposed equation; for $4^4=256$.

2. Reduce the equation $4x^x = 100x^3$. Ans. $x=5$.
3. Reduce the equation $x^x = 9x$.

TRIGONOMETRY.

SECTION I.

SINES, TANGENTS, SECANTS, &c.

ART. 71. **TRIGONOMETRY** *treats of the relations of the sides and angles of TRIANGLES.* Its first object is, to determine the length of the sides, and the quantity of the angles. In addition to this, from its principles are derived many interesting methods of investigation in the higher branches of analysis, particularly in physical astronomy. Scarcely any department of mathematics is more important, or more extensive in its applications. By trigonometry, the mariner traces his path on the ocean; the geographer determines the latitude and longitude of places, the dimensions and positions of countries, the altitude of mountains, the courses of rivers, &c. and the astronomer calculates the distances and magnitudes of the heavenly bodies, predicts the eclipses of the sun and moon, and measures the progress of light from the stars.

72. Trigonometry is either *plane* or *spherical*. The former treats of triangles bounded by *right lines*; the latter, of triangles bounded by *arcs of circles*.

Divisions of the Circle.

73. In a triangle there are two classes of quantities which are the subjects of inquiry, the *sides* and the *angles*. For the purpose of measuring the latter, a *circle* is introduced.

The periphery of every circle, whether great or small, is supposed to be divided into 360 equal parts called *degrees*, each degree into 60 *minutes*, each minute into 60 *seconds*,

each second into 60 *thirds*, &c. marked with the characters $^{\circ}$, $'$, $''$, &c. Thus $32^{\circ} 24' 13'' 22'''$ is 32 degrees 24 minutes 13 seconds 22 thirds.*

A degree, then, is not a magnitude of a given *length*; but a certain *portion* of the whole circumference of any circle. It is evident, that the 360th part of a large circle is greater, than the same part of a small one. On the other hand, the *number* of degrees, in a small circle, is the same as in a large one.

The fourth part of a circle is called a *quadrant*, and contains 90 degrees.

74. To *measure* an angle, a circle is so described that its centre shall be the angular point, and its periphery shall cut the two lines which include the angle. The *arc* between the two lines is considered a *measure of the angle*, because, by Euc. 33. 6, angles at the centre of a given circle, have the same ratio to each other, as the arcs on which they stand. Thus the arc AB, (Fig. 2.) is a measure of the angle ACB.

It is immaterial what is the size of the circle, provided it cuts the lines which include the angle. Thus the angle ACD (Fig. 4.) is measured by either of the arcs AG, *ag*. For ACD is to ACH, as AG to AH, or as *ag* to *ah*. (Euc. 33. 6.)

75. In the circle ADGH (Fig. 2.) let the two diameters AG and DH be perpendicular to each other. The angles ACD, DCG, GCH, and HCA will be right angles; and the periphery of the circle will be divided into four equal parts, each containing 90 degrees. As a right angle is subtended by an arc of 90° , the angle itself is said to contain 90° . Hence, in two right angles, there are 180° , in four right angles 360° ; and in any other angle, as many degrees, as in the arc by which it is subtended.

76. The sum of the three angles of any triangle being equal to two right angles, (Euc. 32. 1.) is equal to 180° . Hence, there can never be more than one obtuse angle in a triangle. For the sum of two obtuse angles is more than 180° .

77. The *COMPLEMENT* of an arc or an angle, is the difference between the arc or angle and 90 degrees.

The complement of the arc AB (Fig. 2.) is DB; and the complement of the angle ACB is DCB. The complement of the arc BDG is also DB.

* See note D.

The complement of 10° is 80° , of 60° is 30° ,
 of 20° is 70° , of 120° is 30° ,
 of 50° is 40° , of 170° is 80° , &c.

Hence, an acute angle and its complement are always equal to 90° . The angles ACB and DCB are together equal to a right angle. The two acute angles of a right angled triangle are equal to 90° : therefore each is the complement of the other.

78. *The SUPPLEMENT of an arc or an angle is the difference between the arc or angle and 180 degrees.*

The supplement of the arc BDG (Fig. 2.) is AB; and the supplement of the angle BCG is BCA.

The supplement of 10° is 170° , of 120° is 60° ,
 of 80° is 100° , of 150° is 30° , &c.

Hence an angle and its supplement are always equal to 180° . The angles BCA and BCG are together equal to two right angles.

79. Cor. As the three angles of a plane triangle are equal to two right angles, that is, to 180° (Euc. 32. 1.) the sum of any two of them is the supplement of the other. So that the third angle may be found, by subtracting the sum of the other two from 180° . Or the sum of any two may be found, by subtracting the third from 180° .

80. A straight line drawn from the centre of a circle to any part of the periphery, is called a *radius* of the circle. In many calculations, it is convenient to consider the radius, whatever be its length, as a *unit*. (Alg. 510.) To this must be referred the numbers expressing the lengths of other lines. Thus 20 will be twenty times the radius, and 0.75, three fourths of the radius.

Definitions of Sines, Tangents, Secants, &c.

81. To facilitate the calculations in trigonometry, there are drawn, within and about the circle, a number of straight lines, called *Sines, Tangents, Secants, &c.* With these the learner should make himself perfectly familiar.

82. *The SINE of an arc is a straight line drawn from one end of the arc, perpendicular to a diameter which passes through the other end.*

Thus BG (Fig. 3.) is the sine of the arc AG. For BG is a line drawn from the end G of the arc, perpendicular to the

diameter AM which passes through the other end A of the arc.

Cor. The sine is *half the chord of double the arc*. The sine BG is half PG, which is the chord of the arc PAG, double the arc AG.

83. The *VERSED SINE* of an arc is that part of the diameter which is between the sine and the arc.

Thus BA is the versed sine of the arc AG.

84. The *TANGENT* of an arc, is a straight line drawn perpendicular from the extremity of the diameter which passes through one end of the arc, and extended till it meets a line drawn from the centre through the other end.

Thus AD (Fig. 3.) is the tangent of the arc AG.

85. The *SECANT* of an arc, is a straight line drawn from the centre, through one end of the arc, and extended to the tangent which is drawn from the other end.

Thus CD (Fig. 3.) is the secant of the arc AG.

86. In Trigonometry, the terms *tangent* and *secant* have a more limited meaning, than in Geometry. In both, indeed, the tangent *touches* the circle, and the secant *cuts* it. But in Geometry, these lines are of no determinate length; whereas, in Trigonometry, they extend from the diameter to the point in which they intersect each other.

87. The lines just defined are sines, tangents and secants of arcs. BG (Fig. 3.) is the sine of the arc AG. But this arc subtends the angle GCA. BG is then the sine of the arc which subtends the angle GCA. This is more concisely expressed, by saying that BG is the sine of the angle GCA. And universally, the sine, tangent, and secant of an arc, are said to be the sine, tangent, and secant of the angle which stands at the centre of the circle, and is subtended by the arc. Whenever, therefore, the sine, tangent, or secant of an angle is spoken of; we are to suppose a circle to be drawn whose centre is the angular point; and that the lines mentioned belong to that arc of the periphery which subtends the angle.

88. The *sine*, and *tangent* of an acute angle, are *opposite* to the angle. But the *secant* is one of the lines which *include* the angle. Thus the sine BG, and the tangent AD (Fig. 3.) are opposite to the angle DCA. But the secant CD is one of the lines which include the angle.

89. The *sine complement* or *COSINE* of an angle, is the *sine* of the *COMPLEMENT* of that angle. Thus, if the diameter HO (Fig. 3.) be perpendicular to MA, the angle HCG is the

complement of ACG ; (Art. 77.) and LG , or its equal CB , is the sine of HCG . (Art. 82.) It is, therefore, the *cosine* of GCA . On the other hand GB is the sine of GCA , and the cosine of GCH .

So also the *cotangent* of an angle is the tangent of the complement of the angle. Thus HF is the cotangent of GCA . And the *cosecant* of an angle is the secant of the complement of the angle. Thus CF is the cosecant of GCA .

Hence, as in a right angled triangle, one of the acute angles is the complement of the other; (Art. 77.) the sine, tangent, and secant of one of these angles, are the cosine, cotangent, and cosecant of the other.

90. The sine, tangent, and secant of the *supplement* of an angle, are each equal to the sine, tangent, and secant of the angle itself. It will be seen, by applying the definition (Art. 82.) to the figure, that the sine of the obtuse angle GCM is BG , which is also the sine of the acute angle GCA . It should be observed, however, that the sine of an acute angle is *opposite* to it; while the sine of an obtuse angle *falls without* the angle, and is opposite to its supplement. Thus BG , the sine of the angle MCG , is not opposite to MCG , but to its supplement ACG .

The *tangent* of the obtuse angle MCG is MT , or its equal AD , which is also the tangent of ACG . And the *secant* of MCG is CD , which is also the secant of ACG .

91. But the *versed sine* of an angle is *not the same*, as that of its supplement. The versed sine of an acute angle is equal to the *difference* between the cosine and radius. But the versed sine of an obtuse angle is equal to the *sum* of the cosine and radius. Thus the versed sine of ACG is $AB=AC-BC$. (Art. 83.) But the versed sine of MCG is $MB=MC+BC$.

Relations of Sines, Tangents, Secants, &c. to each other.

92. The relations of the sine, tangent, secant, cosine, &c. to each other, are easily derived from the proportions of the sides of similar triangles. (Euc. 4. 6.) In the quadrant ACH , (Fig. 3.) these lines form three similar triangles, viz. ACD , BCG or LCG , and HCF . For, in each of these, there is one right angle, because the sines and tangents are, by definition, perpendicular to AC ; as the cosine and cotangent are to CH . The lines CH , BG , and AD are parallel, because CA makes a right angle with each. (Euc. 27. 1.) For the same

reason, CA, LG, and HF are parallel. The alternate angles GCL, BGC, and the opposite angle CDA are equal; (Euc. 29. 1.) as are also the angles GCB, LGC, and HFC. The triangles ACD, BCG, and HCF are therefore similar.

It should also be observed, that the line BC, between the sine and the centre of the circle, is parallel and equal to the cosine; and that LC, between the cosine and centre, is parallel and equal to the sine; (Euc. 34. 1.) so that one may be taken for the other, in any calculation.

93. From these similar triangles, are derived the following proportions; in which R is put for radius,

<i>sin</i> for sine,	<i>cos</i> for cosine,
<i>tan</i> for tangent,	<i>cot</i> for cotangent,
<i>sec</i> for secant,	<i>cosec</i> for cosecant.

By comparing the triangles CBG and CAD,

1. AC:BC::AD:BG, that is, $R:cos::tan:sin$.
2. CG:CD::BG:AD $R:sec::sin:tan$.
3. CB:CA::CG:CD $cos:R::R:sec$.

$$\text{Therefore } R^2 = cos \times sec.$$

By comparing the triangles CLG and CHF,

4. CH:CL::HF:LG, that is, $R:sin::cot:cos$.
5. CG:CF::LG:HF $R:cosec::cos:cot$.
6. CL:CH::CG:CF $sin:R::R:cosec$.

$$\text{Therefore } R^2 = sin \times cosec.$$

By comparing the triangles CAD and CHF,

7. CH:AD::CF:CD, that is $R:tan::cosec:sec$.
8. CA:HF::CD:CF $R:cot::sec:cosec$.
9. AD:AC::CH:HF $tan:R::R:cot$.

$$\text{Therefore } R^2 = tan \times cot.$$

It will not be necessary for the learner to commit these proportions to memory. But he ought to make himself so familiar with the manner of stating them from the figure, as to be able to explain them, whenever they are referred to.

94. Other relations of the sine, tangent, &c. may be derived from the proposition, that the square of the hypotenuse is equal to the sum of the squares of the perpendicular sides. (Euc. 47. 1.)

In the right angled triangles CBG, CAD, and CHF, (Fig. 3.)

1. $\overline{CG}^2 = \overline{CB}^2 + \overline{BG}^2$, that is, $R^2 = \cos^2 + \sin^2$,* ~~X~~
 2. $\overline{CD}^2 = \overline{CA}^2 + \overline{AD}^2$ $\sec^2 = R^2 + \tan^2$,
 3. $\overline{CF}^2 = \overline{CH}^2 + \overline{HF}^2$ $\csc^2 = R^2 + \cot^2$.

And, extracting the root of both sides, (Alg. 296.)

$$R = \sqrt{\cos^2 + \sin^2}, \text{ \&c.}$$

95. $\left. \begin{array}{l} \text{The sine of } 90^\circ \\ \text{The chord of } 60^\circ \\ \text{And the tangent of } 45^\circ \end{array} \right\} \text{ are, in any circle, each equal to the}$
radius, and therefore equal to each other.

Demonstration.

1. In the quadrant ACH, (Fig. 5.) the arc AH is 90° . The sine of this, according to the definition, (Art. 82.) is CH, the radius of the circle.

2. Let AS be an arc of 60° . Then the angle ACS, being measured by this arc, will also contain 60° ; (Art. 75.) and the triangle ACS will be equilateral. For the sum of the three angles is equal to 180° . (Art. 76.) From this, taking the angle ACS which is 60° , the sum of the remaining two is 120° . But these two are *equal*, because they are subtended by the equal sides CA and CS, both radii of the circle. Each, therefore, is equal to *half* 120° , that is, to 60° . All the angles being equal, the sides are equal, and therefore AS, the chord of 60° , is equal to CS the radius.

3. Let AR be an arc of 45° . AD will be its tangent, and the angle ACD subtended by the arc, will contain 45° . The angle CAD is a right angle, because the tangent is, by definition, perpendicular to the radius AC. (Art. 84.) Subtracting ACD, which is 45° , from 90° , (Art. 77.) the other acute angle ADC will be 45° also. Therefore the two legs of the triangle ACD are equal, because they are subtended by equal angles; (Euc. 6. 1.) that is, AD the tangent of 45° , is equal to AC the radius.

Cor. The *cotangent* of 45° is also equal to radius. For the complement of 45° is itself 45° . Thus HD, the cotangent of ACD, (Fig. 5.) is equal to AC the radius.

96. The sine of 30° is equal to *half radius*. For the sine of 30° is equal to half the chord of 60° . (Art. 82. cor.)

* Sine^2 is here put for the square of the sine, $\cos^2$ for the square of the cosine, &c.

But, by the preceding article, the chord of 60° is equal to radius. Its half, therefore, which is the sine of 30° is equal to half radius.

Cor. The *cosine* of 60° is equal to half radius. For the cosine of 60° is the sine of 30° . (Art. 89.)

97. The *chord* of any arc is a *mean proportional*, between the *diameter* of the circle, and the *versed sine* of the arc.

Let ADB (Fig. 6.) be an arc, of which AB is the chord, BF the sine, and AF the versed sine. The angle ABH is a right angle, (Euc. 31. 3.) and the triangles ABH and ABE are similar. (Euc. 8. 6.) Therefore,

$$AH : AB :: AB : AF.$$

That is, the diameter is to the chord, as the chord to the versed sine.*

* See note E.

SECTION II.

THE TRIGONOMETRICAL TABLES.

ART. 98. **T**O facilitate the operations in trigonometry, the sine, tangent, secant, &c. have been calculated, for every degree and minute, and, in some instances, for every second, of a quadrant, and arranged in tables. These constitute what is called the *Trigonometrical Canon*.^{*} It is not necessary to extend these tables beyond 90° ; because the sines, tangents, and secants, are of the same magnitude, in one of the quadrants of a circle, as in the others. Thus the sine of 30° is equal to that of 150° . (Art. 90.)

99. And in any instance, if we have occasion for the sine, tangent, or secant of an *obtuse angle*, we may obtain it, by looking for its equal, the sine, tangent, or secant of the *supplementary acute angle*.

100. The tables are calculated for a circle whose radius is supposed to be a *unit*. It may be an inch, a yard, a mile, or any other denomination of length. But the *sines*, *tangents*, &c. must always be understood to be of the *same* denomination as the radius.

101. All the *sines*, except that of 90° , are *less than the radius*, (Art. 82, and Fig. 3.) and are expressed in the tables by decimals.

Thus the sine of 20° is 0.34202, of 60° is 0.86603,
of 40° is 0.64279, of 89° is 0.99985, &c.

When the tables are intended to be very exact, the decimal is carried to a greater number of places.

The *tangents* of all angles less than 45° are also less than radius. (Art. 95.) But the tangents of angles greater than 45° , are *greater than radius*, and are expressed by a whole number and a decimal. It is evident that all the *secants* also

^{*} For the construction of the Canon, see one of the following sections.

must be greater than radius, as they extend from the centre, to a point without the circle.

102. The numbers in the tables here spoken of, are called *natural* sines, tangents, &c. They express the lengths of the several lines which have been defined in arts. 82, 83, &c. By means of them, the angles and sides of triangles may be accurately determined. But the calculations must be made by the tedious processes of multiplication and division. To avoid this inconvenience, another set of tables has been provided, in which are inserted the *logarithms* of the natural sines, tangents, &c. By the use of these, addition and subtraction are made to perform the office of multiplication and division. On this account, the tables of logarithmic, or as they are sometimes called, *artificial* sines, tangents, &c. are much more valuable, for practical purposes, than the *natural* sines, &c. Still it must be remembered, that the former are derived from the latter. The artificial sine of an angle, is the logarithm of the natural sine of that angle. The artificial tangent is the logarithm of the natural tangent, &c.

103. One circumstance, however, is to be attended to, in comparing the two sets of tables. The radius to which the *natural* sines, &c. are calculated, is *unity*. (Art. 100.) The secants, and a part of the tangents are, therefore, *greater* than a unit; while the sines, and another part of the tangents, are *less* than a unit. When the logarithms of these are taken, some of the indices will be *positive*, and others *negative*; (Art. 9.) and the throwing of them together in the same table, if it does not lead to error, will at least be attended with inconvenience. To remedy this, 10 is added to each of the indices. (Art. 12.) They are then all positive. Thus the natural sine of 20° is 0.34202. The logarithm of this is $\overline{1.53405}$. But the index, by the addition of 10, becomes $10 - 1 = 9$. The logarithmic sine in the tables is therefore 9.53405.*

Directions for taking Sines, Cosines, &c. from the tables.

104. The *cosine*, *cotangent*, and *cosecant* of an angle, are the sine, tangent, and secant of the *complement* of the angle. (Art. 89.) As the complement of an angle is the difference between the angle and 90° , and as 45 is the half of 90; if any given angle within the quadrant is *greater* than 45° , its

* Or the tables may be supposed to be calculated to the radius 10000000000, whose logarithm is 10.

complement is less ; and, on the other hand, if the angle is less than 45° , its complement is greater. Hence, every cosine, cotangent, and cosecant of an angle greater than 45° , has its equal, among the sines, tangents, and secants of angles less than 45° , and *v. v.*

Now, to bring the trigonometrical tables within a small compass, the same column is made to answer for the *sines* of a number of angles *above* 45° , and for the *cosines* of an equal number *below* 45° .

Thus 9.23967 is the log. *sine* of 10° , and the *cosine* of 80° ,
 9.53405 the *sine* of 20° , and the *cosine* of 70° , &c.

The tangents and secants are arranged in a similar manner. Hence,

105. *To find the Sine, Cosine, Tangent, &c. of any number of degrees and minutes.*

If the given angle is less than 45° , look for the degrees at the *top* of the table, and the minutes on the *left* ; then, opposite to the minutes, and under the word *sine* at the head of the column, will be found the *sine* ; under the word *tangent*, will be found the *tangent* ; &c.

The log. sin of $43^\circ 25'$ is 9.83715	The tan of $17^\circ 20'$ is 9.49430
of $17^\circ 20'$ 9.47411	of $8^\circ 46'$ 9.18812
The cos of $17^\circ 20'$ 9.97982	The cot of $17^\circ 20'$ 10.50570
of $8^\circ 46'$ 9.99490	of $8^\circ 46'$ 10.81188

The first figure is the index ; and the other figures are the decimal part of the logarithm.

106. If the given angle is between 45° and 90° ; look for the degrees at the *bottom* of the table, and the minutes on the *right* ; then, opposite to the minutes, and *over* the word *sine* at the foot of the column, will be found the *sine* ; over the word *tangent*, will be found the *tangent*, &c.

Particular care must be taken, when the angle is less than 45° , to look for the title of the column, at the *top*, and for the minutes, on the *left* ; but when the angle is between 45° and 90° , to look for the title of the column, at the *bottom*, and for the minutes, on the *right*.

The log. sine	of $61^\circ 21'$	is 9.99503
The cosine	of $72^\circ 10'$	9.48607
The tangent	of $54^\circ 40'$	10.14941
The cotangent	of $63^\circ 22'$	9.70026

107. If the given angle is *greater* than 90° , look for the sine, tangent, &c. of its *supplement*. (Art. 98, 99.)

The log. sine of	$96^\circ 44'$	is	9.99699
The cosine of	$171^\circ 46'$		9.99494
The tangent of	$130^\circ 26'$		10.06952
The cotangent of	$156^\circ 22'$		10.35894

108. To find the sine, cosine, tangent, &c. of any number of degrees, minutes, and SECONDS.

In the common tables, the sine, tangent, &c. are given, only to every *minute* of a degree.* But they may be found to *seconds*, by taking *proportional parts* of the difference of the numbers as they stand in the tables. For, within a single minute, the variations in the sine, tangent, &c. are nearly proportional to the variations in the angle. Hence,

To find the sine, tangent, &c. to seconds: Take out the number corresponding to the given degree and minute; and also that corresponding to the next greater minute, and find their difference. Then state this proportion;

As 60, to the given number of seconds;

So is the difference found, to the correction for the seconds.

This correction, in the case of sines, tangents, and secants, is to be *added* to the number answering to the given degree and minute; but for cosines, cotangents, and cosecants, the correction is to be *subtracted*.

For, as the sines *increase*, the cosines *decrease*.

Ex. 1. What is the logarithmic sine of $14^\circ 43' 10''$?

The sine of $14^\circ 43'$	is	9.40490
of $14^\circ 44'$		9.40538

Difference	48
------------	----

Here it is evident, that the sine of the required angle is greater than that of $14^\circ 43'$, but less than that of $14^\circ 44'$.

And as the difference corresponding to a whole minute or $60''$ is 48; the difference for $10''$ must be a proportional part of 48. That is,

$60'' : 10'' :: 48 : 8$ the correction to be *added* to the sine of $14^\circ 43'$.

* In the very valuable tables of Michael Taylor, the sines and tangents are given to *every second*.

Therefore the sine of $14^{\circ} 43' 10''$ is 9.40498

2. What is the logarithmic cosine of $32^{\circ} 16' 45''$?

The cosine of $32^{\circ} 16'$ is 9.92715
of $32^{\circ} 17'$ 9.92707

Difference	8
------------	---

Then $60'' : 45'' :: 8 : 6$ the correction to be subtracted from the cosine of $32^{\circ} 16'$.

Therefore the cosine of $32^{\circ} 16' 45''$ is 9.92709

The tangent of $24^{\circ} 15' 18''$ is 9.65376

The cotangent of $31^{\circ} 50' 5''$ is 10.20700

The sine of $58^{\circ} 14' 32''$ is 9.92956

The cosine of $55^{\circ} 10' 26''$ is 9.75670

If the given number of seconds be any even part of 60, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. the correction may be found, by taking a like part of the difference of the numbers in the tables, without stating a proportion in form.

109. To find the degrees and minutes belonging to any given sine, tangent, &c.

This is reversing the method of finding the sine, tangent, &c. (Art. 105, 6, 7.)

Look in the column of the same name, for the sine, tangent, &c. which is *nearest* to the given one; and if the title be at the *head* of the column, take the degrees at the *top* of the table, and the minutes on the *left*; but if the title be at the *foot* of the column, take the degrees at the *bottom*, and the minutes on the *right*.

Ex. 1. What is the number of degrees and minutes belonging to the logarithmic sine 9.62863?

The nearest sine in the tables is 9.62865. The title of sine is at the head of the column in which these numbers are found. The degrees at the top of the page are 25, and the minutes on the left are 10. The angle required is, therefore, $25^{\circ} 10'$

The angle belonging to

the sine 9.87993 is $49^{\circ} 20'$	the cos	9.97351 is $19^{\circ} 48'$
the tan 9.97955 $43^{\circ} 39'$	the cotan	9.75791 $60^{\circ} 12'$
the sec 10.65396 $77^{\circ} 11'$	the cosec	10.49066 $18^{\circ} 51'$

110. To find the degrees, minutes, and seconds belonging to any given sine, tangent, &c.

This is reversing the method of finding the sine, tangent, &c. to seconds. (Art. 108.)

First find the difference between the sine, tangent, &c. next greater than the given one, and that which is next less; then the difference between this less number and the given one; Then

As the difference first found, is to the other difference;

So are 60 seconds, to the number of seconds, which, in the case of sines, tangents, and secants, are to be *added* to the degrees and minutes belonging to the least of the two numbers taken from the tables; but for cosines, cotangents, and cosecants, are to be *subtracted*.

Ex. 1. What are the degrees, minutes, and seconds, belonging to the logarithmic sine 9.40498?

Sine next greater	14° 44'	9.40538	Given sine	9.40498
Next less	14° 43'	9.40490	Next less	9.40490
Difference		<u>48</u>	Difference	<u>8</u>

Then $48 : 8 :: 60'' : 10''$, which added to $14^\circ 43'$ gives $14^\circ 43' 10''$ for the answer.

2. What is the angle belonging to the cosine 9.09773?

Cosine next greater	82° 48'	9.09807	Given cosine	9.09773
Next less	82° 49'	9.09707	Next less	9.09707
Difference		<u>100</u>	Difference	<u>66</u>

Then $100 : 66 :: 60'' : 40''$, which subtracted from $82^\circ 49'$, gives $82^\circ 48' 20''$ for the answer.

It must be observed here, as in all other cases, that of the two angles, the less has the greater cosine.

The angle belonging to

the sin 9.20621 is $9^\circ 15' 6''$ the tan 10.43434 is $69^\circ 48' 16''$
the cos 9.93157 is $16^\circ 34' 30''$ the cot 10.33554 is $24^\circ 47' 16''$

Method of supplying the Secants and Cosecants.

111. In some trigonometrical tables, the secants and co-

secants are not inserted. But they may be easily obtained from the sines and cosines. For, by art. 93, proportion 3d,

$$\cos \times \sec = R^2$$

That is, the product of the cosine and secant, is equal to the square of radius. But, in logarithms, addition takes the place of multiplication; and, in the tables of logarithmic sines, tangents, &c. the radius is 10. (Art. 103.) Therefore, in these tables,

$$\cos + \sec = 20. \quad \text{Or } \sec = 20 - \cos.$$

Again, by art. 93, proportion 6,

$$\sin \times \operatorname{cosec} = R^2.$$

Therefore, in the tables,

$$\sin + \operatorname{cosec} = 20, \quad \text{Or } \operatorname{cosec} = 20 - \sin. \quad \text{Hence,}$$

112. To obtain the *secant*, subtract the cosine from 20; and to obtain the *cosecant*, subtract the sine from 20.

These subtractions are most easily performed, by taking the right hand figure from 10, and the others from 9, as in finding the arithmetical complement of a logarithm; (Art. 55.) observing, however, to add 10 to the index of the secant or cosecant. In fact, the secant is the arithmetical complement of the cosine, with 10 added to the index.

$$\text{For the secant} \quad = 20 - \cos$$

$$\text{And the ar. comp. of } \cos = 10 - \cos \text{ (Art. 54.)}$$

So also the cosecant is the arithmetical complement of the sine, with 10 added to the index. The tables of secants and cosecants are, therefore, of use, in furnishing the arithmetical complement of the sine and cosine, in the following simple manner;

113. For the arithmetical complement of the *sine*, subtract 10 from the index of the cosecant; and for the arithmetical complement of the *cosine*, subtract 10 from the index of the secant.

By this, we may save the trouble of taking each of the figures from 9.

X

SECTION III.

SOLUTIONS OF RIGHT ANGLED TRIANGLES.

ART. 114. **I**N a triangle, there are *six parts*, three sides, and three angles. In every trigonometrical calculation, it is necessary that some of these should be known, to enable us to find the others. *The number of parts which must be given is THREE, one of which must be a SIDE.*

If only two parts be given, they will be either two sides, a side and an angle, or two angles; neither of which will limit the triangle to a particular form and size.

If *two sides* only be given, they may make any angle with each other; and may, therefore, be the sides of a thousand different triangles. Thus the two lines *a* and *b* (Fig. 7.) may belong either to the triangle ABC, or ABC', or ABC''. So that it will be impossible, from knowing two of the sides of a triangle, to determine the other parts.

Or, if a *side and an angle* only be given, the triangle will be indeterminate. Thus, if the side AB (Fig. 8.) and the angle at A be given; they may be parts either of the triangle ABC, or ABC', or ABC''.

Lastly, if *two angles*, or even if *all* the angles be given, they will not determine the length of the sides. For the triangles ABC, A'B'C', A''B''C'', and a hundred others which might be drawn, with sides parallel to these, will all have the same angles. So that one of the parts given must always be a side. If this and any other two parts either sides or angles be known, the other three may be found, as will be shown, in this and the following section.

115. Triangles are either *right angled* or *oblique angled*. The calculations of the former are the most simple, and those which we have the most frequent occasion to make. A great portion of the problems in the mensuration of heights and distances, in surveying, navigation and astronomy, are solved by rectangular trigonometry. Any triangle whatever may be divided into two right angled triangles, by drawing a perpendicular from one of the angles to the opposite side.

116. One of the six parts in a right angled triangle, is always given, viz. the right angle. This is a *constant* quantity; while the other angles and the sides are variable. It is also to be observed, that, if one of the *acute* angles is given, the other is known of course. For one is the complement of the other. (Art. 76, 77.) So that, in a right angled triangle, subtracting one of the acute angles from 90° gives the other. There remain then, only *four* parts, one of the acute angles, and the three sides to be sought by calculation. If any *two* of these be given, with the right angle, the others may be found.

117. To illustrate the method of calculation, let a case be supposed in which a right angled triangle CAD (Fig. 10.) has one of its sides equal to the radius to which the trigonometrical tables are adapted.

In the first place, let the *base* of the triangle be equal to the tabular radius. Then, if a circle be described, with this radius, about the angle C as a centre, DA will be the *tangent*, and DC the *secant* of that angle. (Art. 84, 85.) So that the radius, the tangent, and the secant of the angle at C, constitute the three sides of the triangle. The *tangent*, taken from the tables of natural sines, tangents, &c. will be the length of the *perpendicular*; and the *secant* will be the length of the *hypotenuse*. If the tables used be logarithmic, they will give the *logarithms* of the lengths of the two sides.

In the same manner, *any* right angled triangle whatever, whose base is equal to the radius of the tables, will have its other two sides found among the tangents and secants. Thus, if the quadrant AH (Fig. 11.) be divided into portions of 15° each; then, in the triangle

CAD, AD will be the tan, and CD the sec of 15° ,
 In CAD', AD' will be the tan, and CD' the sec of 30° ,
 In CAD'' AD'' will be the tan, and CD'' the sec of 45° ,
 [&c.

118. In the next place, let the *hypotenuse* of a right angled triangle CBF (Fig. 12.) be equal to the radius of the tables. Then, if a circle be described, with the given radius, and about the angle C as a centre; BF will be the *sine*, and BC the *cosine* of that angle. (Art. 82, 89.) Therefore the sine of the angle at C taken from the tables, will be the length of the *perpendicular*, and the cosine will be the length of the *base*.

And any right angled triangle whatever, whose hypothe-

nuse is equal to the tabular radius, will have its other two sides found among the sines and cosines. Thus if the quadrant AH (Fig. 13.) be divided into portions of 15° each, in the points F, F', F'', &c.; then, in the triangle

CBF, FB will be the sin, and CB the cos, of 15° ,
 In CB'F' F'B' will be the sin, and CB' the cos, of 30° ,
 In CB''F'', F''B'' will be the sin, and CB'' the cos, of 45° , &c.

119. By merely turning to the tables, then, we may find the parts of any right angled triangle which has one of its sides equal to the radius of the tables. But for determining the parts of triangles which have not any of their sides equal to the tabular radius, the following proportion is used:

*As the radius of one circle,
 To the radius of any other;
 So is a sine, tangent, or secant, in one,
 To the sine, tangent, or secant, of the same number of
 degrees, in the other.*

In the two concentric circles AHM, ahm, (Fig. 4.) the arcs AG and ag contain the same number of degrees. (Art. 74.) The sines of these arcs are BG and bg, the tangents AD and ad, and the secants CD and cd. The four triangles CAD, CBG, Cad, and Cbg, are similar. For each of them, from the nature of sines and tangents, contains one right angle; the angle at C is common to them all; and the other acute angle in each is the complement of that at C. (Art. 77.) We have, then, the following proportions. (Euc. 4. 8.)

$$1. CG : Cg :: BG : bg.$$

That is, one radius is to the other, as one sine to the other.

$$2. CA : Ca :: DA : da.$$

That is, one radius is to the other, as one tangent to the other.

$$3. CA : Ca :: CD : Cd.$$

That is, one radius is to the other, as one secant to the other.

$$\text{Cor. } BG : bg :: DA : da :: CD : Cd.$$

That is, as the sine in one circle, to the sine in the other; so is the tangent in one, to the tangent in the other; and so is the secant in one, to the secant in the other.

This is a general principle, which may be applied to most trigonometrical calculations. If one of the sides of the proposed triangle be made radius, each of the other sides will

be the sine, tangent, or secant, of an arc described by this radius. Proportions are then stated, between these lines, and the *tabular* radius, sine, tangent, &c.

120. A line is said to be *made radius*, when a circle is described, or supposed to be described, whose semidiameter is equal to the line, and whose centre is at one end of it.

121. In any right angled triangle, if the *HYPOTHENUSE* be *made radius*, one of the legs will be a *SINE* of its opposite angle, and the other leg a *COSINE* of the same angle.

Thus, if to the triangle ABC (Fig. 14.) a circle be applied, whose radius is AC, and whose centre is A, then BC will be the *sine*, and BA the *cosine*, of the angle at A. (Art. 82, 89.)

If, while the same line is radius, the other end C be made the centre, then BA will be the *sine*, and BC the *cosine*, of the angle at C.

122. If either of the LEGS be *made radius*, the other leg will be a *TANGENT* of its opposite angle, and the hypotenuse will be a *SECANT* of the same angle; that is, of the angle between the secant and the radius.

Thus, if the base AB (Fig. 15.) be *made radius*, the centre being at A, BC will be the *tangent*, and AC the *secant*, of the angle at A. (Art. 84, 85.)

But, if the *perpendicular* BC (Fig. 16.) be *made radius*, with the centre at C, then AB will be the *tangent*, and AC the *secant*, of the angle at C.

123. As the side which is the *sine*, *tangent*, or *secant* of one of the acute angles, is the *cosine*, *cotangent*, or *cosecant* of the other; (Art. 89.) the *perpendicular* BC (Fig. 14.) is the *sine* of the angle A and the *cosine* of the angle C; while the base AB is the *sine* of the angle C, and the *cosine* of the angle A.

If the base is *made radius*, as in Fig. 15, the *perpendicular* BC is the *tangent* of the angle A, and the *cotangent* of the angle C; while the *hypotenuse* is the *secant* of the angle A, and the *cosecant* of the angle C.

If the *perpendicular* is *made radius*, as in Fig. 16, the base AB is the *tangent* of the angle C, and the *cotangent* of the angle A; while the *hypotenuse* is the *secant* of the angle C; and the *cosecant* of the angle A.

124. Whenever a right angled triangle is proposed, whose sides or angles are required; a *similar* triangle may be formed, from the sines, tangents, &c. of the tables. (Art. 117, 118.) The parts required are then found, by stating proportions between the similar sides of the two triangles. If the trian-

gle proposed be ABC , (Fig. 17.) another abc may be formed, having the same angles with the first, but differing from it in the length of its sides, so as to correspond with the numbers in the tables. If similar sides be made radius in both, the remaining similar sides will be lines of *the same name*; that is, if the perpendicular in one of the triangles be a *sine*, the perpendicular in the other will be a sine; if the base in one be a *cosine*, the base in the other will be a cosine, &c.

If the *hypotenuse* in each triangle be made radius, as in Fig. 14, the perpendicular bc will be the *tabular sine* of the angle at a ; and the perpendicular BC will be a sine of the equal angle A , in a circle of which AC is radius.

If the *base* in each triangle be made radius, as in Fig. 15, then the perpendicular bc will be the *tabular tangent* of the angle at a ; and BC will be a tangent of the equal angle A , in a circle of which AB is radius, &c.

125. From the relations of the similar sides of these triangles, are derived the two following *theorems*, which are sufficient for calculating the parts of any right angled triangle whatever, when the requisite data are furnished. One is used, when a *side* is to be found; the other, when an *angle* is to be found.

THEOREM I.

126. When a *side* is required;

*As the tabular sine, tangent, &c. of the
same name with the given side,
To the given side;
So is the tabular sine, tangent, &c. of the
same name with the required side,
To the required side.*

It will be readily seen, that this is nothing more than a statement, in general terms, of the proportions between the similar sides of two triangles, one proposed for solution, and the other formed from the numbers in the tables.

Thus if the *hypotenuse* be *given*, and the *base* or *perpendicular* be *required*; then in Fig. 14, where ac is the *tabular radius*, bc the *tabular sine* of a or its equal A , and ab the *tabular sine* of C ; (Art. 124.)

$$\begin{array}{ll} ac : AC :: bc : BC, \text{ that is, } R : AC :: \sin A : BC. \\ ac : AC :: ab : AB & R : AC :: \sin C : AB. \end{array}$$

In Fig. 15, where ab is the tabular radius, ac the tabular secant of A , and bc the tabular tangent of A ;

$$\begin{array}{l} ac : AC :: bc : BC, \text{ that is, } \text{Sec } A : AC :: \text{Tan } A : BC. \\ ac : AC :: ab : AB \qquad \qquad \text{Sec } A : AC :: R : AB. \end{array}$$

In Fig. 16, where bc is the tabular radius, ac the tabular secant of C , and ab the tabular tangent of C ;

$$\begin{array}{l} ac : AC :: bc : BC, \text{ that is, } \text{Sec } C : AC :: R : BC. \\ ac : AC :: ab : AB \qquad \qquad \text{Sec } C : AC :: \text{Tan } C : AB \end{array}$$

THEOREM II.

127. When an *angle* is required;

*As the given side made radius,
To the tabular radius ;
So is another given side,
To the tabular sine, tangent, &c. of the same name.*

Thus if the side made radius, and one other side be given, then in Fig. 14,

$$\begin{array}{l} AC : ac :: BC : bc, \text{ that is, } AC : R :: BC : \text{Sin } A. \\ AC : ac :: AB : ab \qquad \qquad AC : R :: AB : \text{Sin } C. \end{array}$$

In Fig. 15,

$$\begin{array}{l} AB : ab :: BC : bc, \text{ that is, } AB : R :: BC : \text{Tan } A. \\ AB : ab :: AC : ac \qquad \qquad AB : R :: AC : \text{Sec } A. \end{array}$$

In Fig. 16,

$$\begin{array}{l} BC : bc :: AB : ab, \text{ that is, } BC : R :: AB : \text{Tan } C. \\ BC : bc :: AC : ac \qquad \qquad BC : R :: AC : \text{Sec } C. \end{array}$$

It will be observed, that in these theorems, *angles* are not introduced, though they are among the quantities which are either given or required, in the calculation of triangles. But the tabular sines, tangents, &c. may be considered the *representatives* of angles; as one may be found from the other, by merely turning to the tables.

128. In the theorem for finding a *side*, the first term of the proportion is a *tabular number*. But, in the theorem for finding an *angle*, the first term is a *side*. Hence, in applying the proportions to particular cases, this rule is to be observed,

*To find a SIDE, begin with a tabular number,
To find an ANGLE, begin with a side.*

Radius is to be reckoned among the tabular numbers.

13. In the theorem for finding an *angle*, the first term is *any sine or radius*. As in every proportion, the three first terms must be *given* to enable us to find the fourth, it is evident that when this theorem is applied, the side made radius must be *given* also. But, in the theorem for finding a *side*, the *radius* is *unknown*; and the order of the terms should be altered, as follows.

14. *As the sine, so the radius, so the radius, so the sine.*

15. The *radius* is *given* in both cases, to make ratios equal in the proportion: because, in the table, the natural sine, *radius*, or radius is 1, and in the logarithmic table, the *radius* is 10.

16. The proportions in Trigonometry are of the same kind as other simple proportions. The fourth term is *unknown*, as in the rule: Three in arithmetic, by adding 2, we find the fourth, *two terms, and dividing* the sum by 2, we find the *fourth*. This is the mode of calculation when the *rules*, *arithmetic*, *arithmetic*, &c. are used. But the method of *logarithms* is so much more *expedient*, that this method is almost superseded the other method. In practice, *addition* takes the place of *multiplication*, and *subtraction* the place of *division*.

The *proportions* expressing the *engines* of the *sides* of a triangle, are to be taken from the tables of common logarithms. The *engines* of the *cosine*, *sinus*, &c. are found in the tables of *trigonometric* functions. The calculation is then made by *adding* the first terms, and *subtracting* the second, as follows.

17. The logarithmic radius 10, or, as it is written in the tables, 10.00000, is to be *added* and *subtracted*, that the two terms of which it is used, may be considered as, in effect, reduced to one. Thus, if the *radius* is in the first term, we have only to add the other two terms, and *subtract* 10 from the index: for this is *subtracting* the first term. If *radius* occurs in the second term, the first is to be *added* from the third, after its index is increased by 10. In the same manner, if *radius* is in the third term, the first is to be *added* from the second.

18. Every species of right angled triangles may be solved upon the principle, that the sides of similar triangles are *proportional*, according to the two theorems mentioned above. There are, however, some advantages, however, in giving the examples in distinct classes.

There must be given, in a right angled triangle, *two* of the parts, besides the right angle. (Art. 116.) These may be;

1. The hypotenuse and an angle; or
2. The hypotenuse and a leg; or
3. A leg and an angle; or
4. The two legs.

CASE I.

134. Given { The hypotenuse, } to find { The base and
And an angle; } Perpendicular.

Ex. 1. If the hypotenuse AC (Fig. 17.*) be 45 miles, and the angle at A $32^{\circ} 20'$, what is the length of the base AB, and the perpendicular BC?

In this case, as *sides* only are required, *any* side may be made radius. (Art. 130.)

If the hypotenuse be made radius, as in Fig. 14, BC will be the sine of A, and AB the sine of C, or the cosine of A. (Art. 121.) And if *abc* be a similar triangle, whose hypotenuse is equal to the *tabular* radius, *bc* will be the tabular sine of A, and *ab* the tabular sine of C. (Art. 124.)

To find the *perpendicular*, then, by Theorem I, we have this proportion;

$$ac : AC :: bc : BC.$$

$$\text{Or } R : AC :: \sin A : BC.$$

Whenever the terms Radius, Sine, Tangent, &c. occur in a proportion like this, the *tabular* Radius, &c. is to be understood, as in arts. 126, 127.

The numerical calculation, to find the length of BC, may be made, either by *natural* sines, or by *logarithms*. See art. 131.

By natural Sines.

$$1 : 45 :: 0.53484 : 24.068 = BC.$$

Computation by Logarithms.

As Radius		10.00000
To the hypotenuse	45	1.65321
So is the Sine of A	$32^{\circ} 20'$	9.72823
To the perpendicular	24.068	<u>1.38144</u>

* The parts which are given are distinguished by a mark across the line, or at the opening of the angle, and the parts required, by a cipher.

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Here, the logarithms of the second and third terms are added, and from the sum, the first term 10 is subtracted. (Art. 132.) The remainder is the logarithm of $24.068 = BC$.

Subtracting the angle at A from 90° , we have the angle at C = $57^\circ 40'$. (Art. 116.) Then, to find the *base* AB;

$$ac : AC :: ab : AB$$

$$\text{Or } R : AC :: \sin C : AB$$

By natural Sines.

$$1 : 45 :: 0.84495 : 38.023 = AB$$

Computation by Logarithms.

As Radius		10.00000
To the hypotenuse	45	1.65321
So is the Sine of C	$57^\circ 40'$	9.92683
To the base	38.023	1.58004

Both the sides required are now found, by making the hypotenuse radius. The results here obtained may be verified, by making either of the other sides radius.

If the *base* be made radius, as in Fig. 15, the perpendicular will be the *tangent*, and the hypotenuse the *secant* of the angle at A. (Art. 122.) Then,

$$\sec A : AC :: R : AB$$

$$R : AB :: \tan A : BC$$

By making the arithmetical calculations, in these two proportions, the values of AB and BC will be found the same as before.

If the *perpendicular* be made radius, as in Fig. 16, AB will be the *tangent*, and AC the *secant* of the angle at C. Then,

$$\sec C : AC :: R : BC$$

$$R : BC :: \tan C : AB$$

Ex. 2. If the hypotenuse of a right angled triangle be 250 rods, and the angle at the base $46^\circ 30'$; what is the length of the base and perpendicular?

Ans. The base is 172.1 rods, and the perpendicular 161.35.

CASE II.

135. Given { The hypotenuse, } to find { The angles and
And one leg } The other leg.

Ex. 1. If the hypotenuse (Fig. 18.) be 35 leagues, and the base 26; what is the length of the perpendicular, and the quantity of each of the acute angles?

To find the angles it is necessary that one of the *given* sides be made radius. (Art. 130.)

If the *hypotenuse* be radius, the base and perpendicular will be sines of their opposite angles. Then, by Theorem II;

$$AC : R :: AB : \sin C$$

As the hypotenuse	35	1.54407
To radius		10.00000
So is the base	26	1.41497
To the Sine of C		47° 58½'
		9.87090

Here, the first term is subtracted from the third, after the index is increased by 10. (Art. 132.) The remainder is the logarithmic sine of 47° 58½', which subtracted from 90°, gives the other acute angle 42° 1½'.

Then to find the *perpendicular* by Theorem I;

$$R : AC :: \sin A : BC$$

As Radius		10.00000
To the hypotenuse	35	1.54407
So is the Sine of A	42° 1½'	9.82572
To the perpendicular		23.43
		1.36979

If the *base* be radius, the perpendicular will be *tangent*, and the hypotenuse *secant* of the angle at A. Then,

$$AB : R :: AC : \sec A$$

$$R : AB :: \tan A : BC$$

In this example, where the hypotenuse and base are given, the angles can not be found by making the *perpendicular* radius. For to find an angle, a *given* side must be made radius. (Art. 130.)

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136. Ex. 2. If the hypotenuse (Fig. 19.) be 54 miles, and the perpendicular 48 miles, what are the angles, and the base?

Making the *hypotenuse* radius.

$$AC : R :: BC : \sin A$$

$$R : AC :: \sin C : AB$$

The numerical calculation will give $A = 62^\circ 44'$, and $AB = 24.74$.

Making the *perpendicular* radius.

$$BC : R :: AC : \sec C$$

$$R : BC :: \tan C : AB$$

The angles can not be found by making the *base* radius, when its length is not given.

CASE III.

137. Given $\left\{ \begin{array}{l} \text{The angles,} \\ \text{And one leg} \end{array} \right\}$ to find $\left\{ \begin{array}{l} \text{The hypotenuse,} \\ \text{And the other leg.} \end{array} \right\}$

Ex. 1. If the base (Fig. 20,) be 60, and the angle at the base $47^\circ 12'$, what is the length of the hypotenuse and the perpendicular?

In this case, as *sides* only are required, *any* side may be radius.

Making the *hypotenuse* radius.

$$\sin C : AB :: R : AC$$

As the Sine of C	$42^\circ 48'$	9.83215
To the base	60	1.77815
So is Radius		10.
To the hypotenuse	88.31	1.94600

To find the *perpendicular*.

$$R : AC :: \sin A : BC$$

As Radius		10.
To the hypotenuse	88.31	1.94600
So is the Sine of A	$47^\circ 12'$	9.86554
To the perpendicular	64.8	1.81154

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Making the *base* radius.

$$R : AB :: \sec A : AC$$

$$R : AB :: \tan A : BC$$

Making the *perpendicular* radius.

$$\tan C : AB :: R : BC$$

$$R : BC :: \sec C : AC$$

138. Ex. 2. If the perpendicular (Fig. 21.) be 74, and the angle C $61^{\circ} 27'$, what is the length of the base and the hypotenuse?

Making the *hypotenuse* radius.

$$\sin A : BC :: R : AC$$

$$R : AC :: \sin C : AB$$

Making the *base* radius.

$$\tan A : BC :: R : AB$$

$$R : AB :: \sec A : AC$$

Making the *perpendicular* radius.

$$R : BC :: \sec C : AC$$

$$R : BC :: \tan C : AB$$

The hypotenuse is 154.83 and the base 136.

CASE IV.

139. Given { The base, and } to find { The hypotenuse,
Perpendicular } And the angles.

Ex. 1. If the base (Fig. 22,) be 284, and the perpendicular 192, what are the angles, and the hypotenuse?

In this case, one of the legs must be made radius, to find an angle; because the hypotenuse is not given.

Making the *base* radius.

$$AB : R :: BC : \tan A$$

As the base	284	2.45332
To radius		10.
So is the perpendicular	192	2.28330
To the tangent of A	$34^{\circ} 4'$	<u>9.82996</u>

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To find the hypotenuse.

$$R : AB :: \sec A : AC$$

As Radius		10.
To the base	284	2.45332
So is the Secant of A	34° 4'	10.08177
To the hypotenuse	342.84	<u>2.53509</u>

Making the *perpendicular* radius.

$$BC : R :: AB : \tan C$$

$$R : BC :: \sec C : AC$$

Ex. 2. If the base be 640, and the perpendicular 480, what are the angles and hypotenuse?

Ans. The hypotenuse is 800, and the angle at the base 36° 52' 12".

Examples for practice.

1. Given the hypotenuse, 68, and the angle at the base 39° 17'; to find the base and perpendicular.
2. Given the hypotenuse 850, and the base 594, to find the angles, and the perpendicular.
3. Given the hypotenuse 78 and perpendicular 57, to find the base, and the angles.
4. Given the base 723, and the angle at the base 64° 18', to find the hypotenuse and perpendicular.
5. Given the perpendicular 632, and the angle at the base 81° 36', to find the hypotenuse and the base.
6. Given the base 32, and the perpendicular 24, to find the hypotenuse, and the angles.*

140. The preceding solutions are all effected, by means of the tabular sines, tangents, and secants. But, when any *two sides* of a right angled triangle are given. the third side may be found, without the aid of the trigonometrical tables, by the proposition, that *the square of the hypotenuse is equal to the sum of the squares of the two perpendicular sides.* (Euc. 47. 1.)

If the legs be given, extracting the square root of the *sum* of their squares, will give the hypotenuse. Or, if the hy-

* See note F.

pothenuse and one leg be given, extracting the square root of the *difference* of the squares, will give the other leg.

Let h = the hypotenuse
 p = the perpendicular
 b = the base } of a right angled triangle.

Then $h^2 = b^2 + p^2$, or (Alg. 296.) $h = \sqrt{b^2 + p^2}$

By transp. $b^2 = h^2 - p^2$, or $b = \sqrt{h^2 - p^2}$

And $p^2 = h^2 - b^2$, or $p = \sqrt{h^2 - b^2}$

Ex. 1. If the base is 32, and the perpendicular 24, what is the hypotenuse?

The square of 32 is 1024
 of 24 is 576

The sum of the squares is 1600
 The root of this sum is 40, the hypotenuse.

2. If the hypotenuse is 100, and the base 80, what is the perpendicular?

The square of 100 is 10000
 of 80 6400

The difference of the squares is 3600
 The root of which is 60, the perpendicular.

3. If the hypotenuse is 300, and the perpendicular 220, what is the base?

Ans. $300^2 - 220^2 = 41600$, the root of which is 204 nearly.

141. It is generally most convenient to find the difference of the squares by *logarithms*. But this is not to be done by *subtraction*. For subtraction, in logarithms, performs the office of *division*. (Art. 41.) If we subtract the logarithm of b^2 from the logarithm of h^2 , we shall have the logarithm, not of the *difference* of the squares, but of their *quotient*. There is, however, an indirect, though very simple method, by which the difference of the squares may be obtained by logarithms. It depends on the principle, that the *difference of the squares of two quantities is equal to*

the product of the sum and difference of the quantities. (Alg. 235.) Thus

$$h^2 - b^2 = (h+b) \times (h-b)$$

as will be seen at once, by performing the multiplication. The two factors may be multiplied by adding their logarithms. Hence,

142. To obtain the difference of the squares of two quantities, add the logarithm of the sum of the quantities, to the logarithm of their difference. After the logarithm of the difference of the squares is found; the square root of this difference is obtained, by dividing the logarithm by 2. (Art. 47.)

Ex. 1. If the hypotenuse be 75 inches, and the base 45, what is the length of the perpendicular?

Sum of the given sides	120	log. 2.07918
Difference of do.	30	1.47712
	Dividing by	2)3.55630
Side required	60	1.77815

2. If the hypotenuse is 135, and the perpendicular 108, what is the length of the base?

Ans. 81.

SECTION IV.



SOLUTIONS OF OBLIQUE ANGLED TRIANGLES.

ART. 143. **T**HE sides and angles of oblique angled triangles may be calculated by the following theorems.

THEOREM I.

In any plane triangle, *the sines of the angles are as their opposite sides.*

Let the angles be denoted by the letters A, B, C, and their opposite sides by a, b, c , as in Fig. 23 and 24. From one of the angles, let the line p be drawn perpendicular to the opposite side. This will fall either within or without the triangle.

1. Let it fall *within* as in Fig. 23. Then, in the right angled triangles ACD and BCD, according to art. 126.

$$R : b :: \sin A : p$$

$$R : a :: \sin B : p.$$

Here, the two *extremes* are the same in both proportions. The other four terms are, therefore, *reciprocally* proportional: (Alg. 387.*) that is,

$$a : b :: \sin A : \sin B.$$

2. Let the perpendicular p fall *without* the triangle, as in Fig. 24. Then, in the right angled triangles ACD and BCD;

$$R : b :: \sin A : p$$

$$R : a :: \sin B : p$$

Therefore as before,

$$a : b :: \sin A : \sin B.$$

Sin A is here put both for the sine of DAC, and for that

*Euclid 23. 5.

of BAC. For, as one of these angles is the *supplement* of the other, they have the same sine. (Art. 90.)

The sines which are mentioned here, and which are used in calculation, are *tabular sines*. But the proportion will be the same, if the sines be adapted to any other radius. (Art. 119.)

THEOREM II.

144. In a plane triangle,

*As the sum of any two of the sides,
To their difference;
So is the tangent of half the sum of the opposite angles,
To the tangent of half their difference.*

Thus the sum of AB and AC (Fig. 25.) is to their difference; as the tangent of half the sum of the angles ACB and ABC, to the tangent of half their difference.

Demonstration.

Extend CA to G, making AG equal to AB; then CG is the sum of the two sides AB and AC. On AB, set off AD equal to AC; then BD is the difference of the sides AB and AC.

The sum of the two angles ACB and ABC, is equal to the sum of ACD and ADC; because each of these sums is the supplement of CAD. (Art. 79.) But, as AC=AD by construction, the angle ADC=ACD. (Euc. 5. 1.) Therefore ACD is half the sum of ACB and ABC. As AB=AG, the angle AGB=ABG or DBE. Also GCE or ACD=ADC=BDE. (Euc. 15. 1.) Therefore, in the triangles GCE and DBE, the two remaining angles DEB and CEG are equal; (Art. 79.) So that CE is perpendicular to BG. (Euc. Def. 10. 1.) If then CE is made radius, GE is the tangent of GCE, (Art. 84.) that is, the tangent of half the sum of the angles opposite to AB and AC.

If from the greater of the two angles ACB and ABC, there be taken ACD their half sum; the remaining angle ECB will be their half difference. (Alg. 341.) The tangent of this angle, CE being radius, is EB, that is, the tangent of half the difference of the angles opposite to AB and AC. We have then,

CG=the sum of the sides AB and AC ;
 DB=their difference ;
 GE=the tangent of half the sum of the opposite angles ;
 EB=the tangent of half their difference.

But, by similar triangles,

$$CG : DB :: GE : EB \quad Q. E. D.$$

THEOREM III.

145. If upon the longest side of a triangle, a perpendicular be drawn from the opposite angle ;

*As the longest side,
 To the sum of the two others ;
 So is the difference of the latter,
 To the difference of the segments made by the
 perpendicular.*

In the triangle ABC (Fig. 26.) if a perpendicular be drawn from C upon AB ;

$$AB : CB + CA :: CB - CA : BP - PA^*$$

Demonstration.

Describe a circle, on the centre C, and with the radius BC. Through A and C, draw the diameter LD, and extend BA to H. Then by Euc. 35, 3,

$$AB \times AH = AL \times AD$$

And converting the equation into a proportion,

$$AB : AD :: AL : AH$$

$$\text{But } AD = CD + CA = CB + CA$$

$$\text{And } AL = CL - CA = CB - CA$$

$$\text{And } AH = HP - PA = BP - PA \text{ (Euc. 3. 3.)}$$

If then, for the three last terms in the proportion, we substitute their equals, we have,

$$AB : CB + CA :: CB - CA : PB - PA$$

146. It is to be observed, that the greater segment is next the greater side. If BC is greater than AC, (Fig. 26.) PB is greater than AP. With the radius AC, describe the arc

* See note G.

AN. The segment $NP = AP$. (Euc. 3. 3.) But BP is greater than NP .

147. The two segments are to each other, as the tangents of the opposite angles, or the cotangents of the adjacent angles. For, in the right angled triangles ACP and BCP , (Fig. 26.) if CP be made radius, (Art. 126.)

$$R : PC :: \tan ACP : AP$$

$$R : PC :: \tan BCP : BP$$

Therefore, by equality of ratios, (Alg. 364.)*

$$\tan ACP : AP :: \tan BCP : BP$$

That is, the segments are as the tangents of the opposite angles. And the tangents of these angles are the *cotangents* of the adjacent angles A and B . (Art. 89.)

Cor. The greater segment is opposite to the greater angle. And of the angles at the base, the less is next the greater side. If BP is greater than AP , the angle BCP is greater than ACP ; and B is less than A . (Art. 77.)

148. To enable us to find the sides and angles of an oblique angled triangle, *three* of them must be *given*. (Art. 114.)

These may be, either

1. Two angles and a side, or
2. Two sides and an angle *opposite* one of them, or
3. Two sides and the *included* angle, or
4. The three sides.

The two first of these cases are solved by theorem I, (Art. 143.) the third by theorem II, (Art. 144.) and the fourth by theorem III, (Art. 145.)

149. In making the calculations, it must be kept in mind, that the greater side is always opposite to the greater angle, (Euc. 18, 19. 1.) that there can be only one *obtuse* angle in a triangle, (Art. 76.) and therefore, that the angles opposite to the two least sides must be *acute*.

* Euc. 11. 5.

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CASE I.

150. Given,
Two angles, and } to find { The remaining angle, and
A side, } { The other two sides.

The third angle is found, by merely subtracting the sum of the two which are given from 180° . (Art. 79.)

The sides are found, by stating, according to theorem I, the following proportion;

As the sine of the angle opposite the *given* side,
To the length of the given side;
So is the sine of the angle opposite the *required* side,
To the length of the required side.

As a *side* is to be found, it is necessary to begin with a *tabular number*.

Ex. 1. In the triangle ABC (Fig. 27.) the side *b* is given 32 rods, the angle A $56^\circ 20'$, and the angle C $49^\circ 10'$, to find the angle B, and the sides *a* and *c*.

The sum of the two given angles $56^\circ 20' + 49^\circ 10' = 105^\circ 30'$; which subtracted from 180° , leaves $74^\circ 30'$ the angle B.

Then,

$$\sin B : b :: \begin{cases} \sin A : a \\ \sin C : c \end{cases}$$

Calculation by logarithms.

As the Sine of B	$74^\circ 30'$	<i>a. c.</i>	0.01609
To the side <i>b</i>	32		1.50515
So is the Sine of A	$56^\circ 20'$		9.92027
To the side <i>a</i>	27.64		1.44151
As the Sine of B	$74^\circ 30'$	<i>a. c.</i>	0.01609
To the side <i>b</i>	32		1.50515
So is the Sine of C	$49^\circ 10'$		9.87887
To the side <i>c</i>	25.13		1.40011

The *arithmetical complement* used in the first term here, may be found, in the usual way, or by taking out the *cosecant* of the given angle, and rejecting 10 from the index. (Art. 113.)

Ex. 2. Given the side b 71, the angle A $107^{\circ} 6'$, and the angle C $27^{\circ} 40'$; to find the angle B , and the sides a and c . The angle B is $45^{\circ} 14'$. Then

$$\sin B : b :: \begin{cases} \sin A : a = 95.58 \\ \sin C : c = 46.43 \end{cases}$$

When one of the given angles is *obtuse*, as in this example, the sine of its *supplement* is to be taken from the tables. (Art. 99.)

CASE II.

151. Given

Two sides, and $\left. \begin{array}{l} \text{An opposite angle,} \end{array} \right\}$ to find $\left\{ \begin{array}{l} \text{The remaining side, and} \\ \text{The other two angles.} \end{array} \right.$

One of the required angles is found, by beginning with a side, and, according to theorem I, stating the proportion,

As the side opposite the given angle,
To the sine of that angle;
So is the side opposite the required angle,
To the sine of that angle.

The third angle is found, by subtracting the sum of the other two from 180° ; and the remaining side is found, by the proportion in the preceding article.

152. In this second case, if the side opposite to the given angle be shorter than the other given side, the solution will be *ambiguous*. Two different triangles may be formed, each of which will satisfy the conditions of the problem.

Let the side b , (Fig. 28.) the angle A , and the length of the side opposite this angle be given. With the latter for radius, (if it be shorter than b ,) describe an arc, cutting the line AH in the points B and B' . The lines BC and $B'C$ will be equal. So that, with the same data, there may be formed two different triangles, ABC and $AB'C$.

There will be the same ambiguity in the numerical calculation. The answer found by the proportion will be the *sine* of an angle. But this may be the sine, either of the *acute* angle $AB'C$, or of the *obtuse* angle ABC . For, BC being equal to $B'C$, the angle $CB'B$ is equal to CBB' . Therefore ABC , which is the supplement of CBB' is also the supplement of $CB'B$. But the sine of an angle is the same, as the sine of its supplement. (Art. 90.) The result of the

calculation will, therefore, be ambiguous. In practice, however, there will generally be some circumstance which will determine whether the angle required is acute or obtuse.

If the side opposite the given angle be *longer* than the other given side, the angle which is subtended by the latter, will necessarily be acute. For there can be but one obtuse angle in a triangle, and this is always subtended by the longest side. (Art. 149.)

If the *given* angle be obtuse, the other two will, of course, be acute. There can, therefore, be no ambiguity in the solution.

Ex. 1: Given the angle A (Fig. 28.) $35^{\circ} 20'$, the opposite side a 50, and the side b 70; to find the remaining side, and the other two angles:

To find the angle opposite to b , (Art. 151.)

$$a : \text{Sin } A :: b : \text{Sin } B$$

As the side a	50	$a. c.$	8.30103
To the Sine of A	$35^{\circ} 20'$		9.76218
So is the side b	70		1.84510

To the Sine of B' or B $54^{\circ} 3' 50''$ or $125^{\circ} 56' 10''$ 9.90831

The calculation here gives the acute angle $AB'C$ $54^{\circ} 3' 50''$, and the obtuse angle ABC $125^{\circ} 56' 10''$. If the latter be added to the angle at A $35^{\circ} 20'$, the sum will be $161^{\circ} 16' 10''$, the supplement of which $18^{\circ} 43' 50''$ is the angle ACB . Then in the triangle ABC , to find the side $c=AB$,

$$\text{Sin } A : a :: \text{Sin } C : c$$

As the Sine of A	$35^{\circ} 20'$	$a. c.$	0.23782
To the side a	50		1.69897
So is the Sine of C	$18^{\circ} 43' 50''$		9.50666
To the side c	27.76		<u>1.44345</u>

If the *acute* angle $AB'C$ $54^{\circ} 3' 50''$ be added to the angle at A $35^{\circ} 20'$, the sum will be $89^{\circ} 23' 50''$, the supplement of which $90^{\circ} 36' 10''$ is the angle ACB' . Then, in the triangle $AB'C$,

$$\text{Sin } A : CB' :: \text{Sin } C : AB' = 86.45$$

K

Ex. 2. Given the angle at A $63^{\circ} 35'$ (Fig. 29.) the side b 64, and the side a 72; to find the side c , and the angles B and C.

$$a : \sin A :: b : \sin B = 52^{\circ} 45' 25''$$

$$\sin A : a :: \sin C : c = 72.05$$

The sum of the angles A and B is $116^{\circ} 20' 25''$, the supplement of which $63^{\circ} 39' 35''$ is the angle C.

In this example the solution is *not ambiguous*, because the side opposite the given angle is longer than the other given side.

Ex. 3. In a triangle of which the angles are A, B, and C, and the opposite sides a , b , and c , as before; if the angle A be $121^{\circ} 40'$, the opposite side a 65 rods, and the side b 47 rods; what are the angles B and C, and what is the length of the side c ? Ans. B is $36^{\circ} 2' 4''$, C $22^{\circ} 17' 56''$, and c 30.3.

In this example also, the solution is not ambiguous, because the *given angle* is obtuse.

CASE III.

153. Given

Two sides, and $\left. \begin{array}{l} \text{The included angle,} \end{array} \right\}$ to find $\left\{ \begin{array}{l} \text{The remaining side, and} \\ \text{The other two angles.} \end{array} \right.$

In this case, the angles are found by theorem II. (Art. 144.) The required side may be found by theorem I.

In making the solutions; it will be necessary to observe, that by subtracting the given angle from 180° , the sum of the other two angles is found; (Art. 79.) and, that *adding half the difference of two quantities to their half sum gives the greater quantity, and subtracting the half difference from the half sum gives the less.* (Alg. 341.) The latter proposition may be geometrically demonstrated thus;

Let AE (Fig. 32.) be the greater of two magnitudes, and BE the less. Bisect AB in D, and make AC equal to BE. Then

AB is the *sum* of the two magnitudes;

CE their *difference*;

DA or DB *half* their *sum*;

DE or DC *half* their *difference*;

But $DA + DE = AE$ the *greater* magnitude;

And $DB - DE = BE$ the *less*.

TRIANGLES.

75

Ex. 1. In the triangle ABC (Fig. 30.) the angle A is given $26^{\circ} 14'$, the side b 39, and the side c 53; to find the angles B and C, and the side a .

The *sum* of the sides b and c is $53+39=92$,
 And their *difference* $53-39=14$.
 The sum of the angles B and C $=180^{\circ}-26^{\circ} 14'=153^{\circ} 46'$
 And *half* the sum of B and C is $76^{\circ} 53'$

Then, by theorem II,

$$(b+c):(b-c)::\tan \frac{1}{2}(B+C):\tan \frac{1}{2}(B-C)$$

As the sum of the sides	$(b+c)$ 92	$a. c.$ 8.03621
To their difference	$(b-c)$ 14	1.14613
So is the tangent of	$\frac{1}{2}(B+C)$ $76^{\circ} 53'$	10.63262
To the tangent of	$\frac{1}{2}(B-C)$ $33^{\circ} 8' 50''$	9.81496

To and from the half sum	$76^{\circ} 53'$
Adding and subtracting the half difference	33 8 50
We have the greater angle	110 1 50
And the less angle	43 44 10

As the greater of the two given sides is c , the greater angle is C, and the less angle B. (Art. 149.)

To find the side a , by theorem I,

$$\sin B:b::\sin A:a$$

As the Sine of B	$43^{\circ} 44' 10''$	$a. c.$ 0.16031
To the side b	39	1.59106
So is the Sine of A	$26^{\circ} 14'$	9.64545
To the side a	24.94	1.39682

Ex. 2. Given the angle A $101^{\circ} 30'$ the side b 76, and the side c 109; to find the angles B and C, and the side a .

B is $30^{\circ} 57\frac{1}{2}'$, C $47^{\circ} 32\frac{1}{2}'$, and a 144.8.

CASE IV.

154. Given the three sides, to find the angles.

In this case, the solutions may be made, by drawing a perpendicular to the longest side, from the opposite angle. This will divide the given triangle into two *right angled* triangles. The two segments may be found by theorem III. (Art. 145.) There will then be given, in each of the right angled triangles, the hypotenuse and one of the legs, from which the angles may be determined, by rectangular trigonometry. (Art. 135.)

Ex. 1. In the triangle ABC (Fig. 31.) the side AB is 39, AC 35, and BC 27. What are the angles?

Let a perpendicular be drawn from C, dividing the longest side AB into the two segments AP and BP. Then by theorem III,

$$AB : AC + BC :: AC - BC : AP - BP$$

As the longest side	39	a. c.	8.40894
To the sum of the two others	62		1.79239
So is the difference of the latter	8		0.90309

To the difference of the segments	12.72		1.10442
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The greater of the two segments is AP, because it is next the side AC, which is greater than BC. (Art. 146.)

To and from half the sum of the segments	19.5
Adding and subtracting half their difference, (Art. 153.)	6.36

We have the greater segment AP	25.86
And the less BP	13.14

Then, in each of the right angled triangles APC and BPC, we have given the hypotenuse and base; and by art. 135,

$$AC : R :: AP : \cos A = 42^\circ 21' 57''$$

$$BC : R :: BP : \cos B = 60^\circ 52' 42''$$

And subtracting the sum of the angles A and B from 180° , we have the remaining angle $ACB = 76^\circ 45' 21''$.

Ex. 2. If the three sides of a triangle are 78, 96, and 104; what are the angles?

Ans. $45^\circ 41' 48''$, $61^\circ 43' 27''$, and $72^\circ 34' 45''$.

* See note H.

Examples for Practice.

1. Given the angle A $54^{\circ} 30'$, the angle B $63^{\circ} 10'$, and the side a 164 rods; to find the angle C, and the sides b and c .
2. Given the angle A $45^{\circ} 6'$ the opposite side a 93, and the side b 108; to find the angles B and C, and the side c .
3. Given the angle A $67^{\circ} 24'$, the opposite side a 62, and the side b 46; to find the angles B and C, and the side c .
4. Given the angle A $127^{\circ} 42'$, the opposite side a 381, and the side b 184; to find the angles B and C, and the side c .
5. Given the side b 58, the side c 67, and the included angle A $= 36^{\circ}$; to find the angles B and C, and the side a .
6. Given the three sides, 631, 268, and 546; to find the angles.

155. The three theorems demonstrated in this section, have been here applied to *oblique angled* triangles only. But they are equally applicable to *right angled* triangles.

Thus, in the triangle ABC, (Fig. 17.) according to theorem I, (Art. 143.)

$$\sin B : AC :: \sin A : BC$$

This is the same proportion as one stated in art. 134, except that, in the first term here, the *sine of B* is substituted for *radius*. But, as B is a right angle, its sine is *equal* to radius. (Art. 95.)

Again, in the triangle ABC, (Fig. 21.) by the same theorem;

$$\sin A : BC :: \sin C : AB$$

This is also one of the proportions in rectangular trigonometry, when the hypotenuse is made radius.

The other two theorems might be applied to the solution of right angled triangles. But, when one of the angles is *known* to be a right angle, the methods explained in the preceding section, are much more simple in practice.*

* For the application of Trigonometry to the Mensuration of Heights and Distances, see Navigation and Surveying.

SECTION V.

GEOMETRICAL CONSTRUCTION OF TRIANGLES, BY THE PLANE SCALE.

ART. 156. **T**O facilitate the construction of geometrical figures, a number of graduated lines are put upon the common two feet scale; one side of which is called the *Plane Scale*, and the other side, *Gunter's Scale*. The most important of these are the scales of *equal parts*, and the line of *chords*. In forming a given triangle, or any other right lined figure, the parts which must be made to agree with the conditions proposed, are the *lines*, and the *angles*. For the former, a scale of equal parts is used; for the latter, a line of chords.

157. The line on the upper side of the plane scale, is divided into *inches* and *tenths* of an inch. Beneath this, on the left hand, are two *diagonal* scales of equal parts,* divided into inches and half inches, by perpendicular lines. On the larger scale, one of the inches is divided into tenths, by lines which pass *obliquely* across, so as to intersect the parallel lines which run from right to left. The use of the oblique lines is to measure *hundredths* of an inch, by inclining more and more to the right, as they cross each of the parallels.

To take off, for instance, an extent of 3 inches, 4 tenths, and 6 hundredths;

Place one foot of the compasses at the intersection of the perpendicular line marked 3 with the parallel line marked 6, and the other foot at the intersection of the latter with the oblique line marked 4.

The other diagonal scale is of the same nature. The divisions are smaller, and are numbered from left to right.

158. In geometrical constructions, what is often required, is to make a figure, not *equal* to a given one, but only *similar*. Now figures are similar which have equal angles, and the

* These lines are not represented in the plate, as the learner is supposed to have the scale before him.

sides about the equal angles *proportional*. (Euc. Def. 1. 6.) Thus a land surveyor, in plotting a field, makes the several lines in his plan to have the same proportion to each other, as the sides of the field. For this purpose, a scale of equal parts may be used, of any dimensions whatever. If the sides of the field are 2, 5, 7, and 10 *rods*, and the lines in the plan are 2, 5, 7, and 10 *inches*, and if the angles are the same in each, the figures are similar. One is a copy of the other, upon a smaller scale.

So any two right lined figures are similar, if the angles are the same in both, and if the number of smaller parts in each side of one, is equal to the number of larger parts in the corresponding sides of the other. The several divisions on the scale of equal parts may, therefore, be considered as representing any measures of length, as feet, rods, miles, &c. All that is necessary is, that the scale be not changed, in the construction of the same figure; and that the several divisions and subdivisions be properly proportioned to each other. If the larger divisions, on the diagonal scale, are units, the smaller ones are tenths and hundredths. If the larger are tens, the smaller are units and tenths.

159. In laying down an *angle*, of a given number of degrees, it is necessary to *measure* it. Now the proper measure of an angle is an arc of a circle. (Art. 74.) And the measure of an arc, where the radius is given, is its *chord*. For the chord is the distance, in a straight line, from one end of the arc to the other. Thus the chord AB (Fig. 33.) is a measure of the arc ADB, and of the angle ACB.

To form the *line of chords*, a circle is described, and the lengths of its chords determined for every degree of the quadrant. These measures are put on the plane scale, on the line marked CHO.

160. The chord of 60° is equal to *radius*. (Art. 95.) In laying down or measuring an angle, therefore, an arc must be drawn, with a radius which is equal to the extent from 0 to 60 on the line of chords. There are generally, on the scale, two lines of chords. Either of these may be used; but the angle must be measured by the same line from which the radius is taken.

161. To *make an angle*, then, of a given number of degrees; From one end of a straight line as a centre, and with a radius equal to the chord of 60° on the line of chords, describe an arc of a circle cutting the straight line. From the point of intersection, extend the chord of the given number

of degrees, applying the other extremity to the arc; and through the place of meeting, draw the other line from the angular point.

If the given angle is *obtuse*, take from the scale the chord of *half* the number of degrees, and apply it *twice* to the arc. Or make use of the chords of any two arcs whose *sum* is equal to the given number of degrees.

A *right angle* may be constructed, by drawing a perpendicular without using the line of chords.

Ex: 1. To make an angle of 32 degrees. (Fig. 33.) With the point C, in the line CH, for a centre, and with the chord of 60° for radius, describe the arc ADF. Extend the chord of 32° from A to B; and through B, draw the line BC. Then is ACB an angle of 32 degrees.

2. To make an angle of 140 degrees. (Fig. 34.) On the line CH, with the chord of 60°, describe the arc ADF; and extend the chord of 70° from A to D, and from D to B. The arc ADB = 70° × 2 = 140°.

On the other hand;

162. To measure an angle; On the angular point as a centre, and with the chord of 60° for radius, describe an arc to cut the two lines which include the angle. The distance between the points of intersection, applied to the line of chords, will give the measure of the angle in degrees. If the angle be *obtuse*, divide the arc into two parts.

Ex. 1. To measure the angle ACB. (Fig. 33.) Describe the arc ADF cutting the lines CH and CB. The distance AB will extend 32° on the line of chords.

2. To measure the angle ACB. (Fig. 34.) Divide the arc ADB into two parts, either equal or unequal, and measure each part, by applying its chord to the scale. The sum of the two will be 140°.

163. Besides the lines of chords, and of equal parts, on the plane scale; there are also lines of natural *sines*, *tangents*, and *secants*, marked Sin. Tan. & Sec. of *semitangents* marked S. T. of *longitude* marked Lon. or M. L. of *Rhumbs* marked Rhu. or Rum. &c. These are not necessary in trigonometrical constructions. Some of them are used in Navigation; and some of them, in the projections of the Sphere.

164. In Navigation, the quadrant, instead of being graduated in the usual manner, is divided into *eight* portions, called *Rhumbs*. The *Rhumb line*, on the scale, is a line of chords, divided into rhumbs and quarter-rhumbs, instead of degrees.

165. The line of *Longitude* is intended to show the num-

ber of geographical miles in a degree of longitude, at different distances from the equator. It is placed over the line of chords, with the numbers in an inverted order: so that the figure above shows the length of a degree of longitude, in any latitude denoted by the figure below*. Thus at the equator, where the latitude is 0, a degree of longitude is 60 geographical miles. In latitude 40, it is 46 miles; in latitude 60, 30 miles, &c.

166. The graduation on the line of *secants* begins where the line of sines ends. For the greatest sine is only equal to radius; but the secant of the least arc is greater than radius.

167. The *semitangents* are the tangents of *half* the given arcs. Thus the semitangent of 20° is the tangent of 10° . The line of semitangents is used in one of the projections of the sphere.



168. In the construction of *triangles*, the sides and angles which are *given* are laid down according to the directions in arts. 158, 161. The parts *required* are then measured, according to arts. 158, 162. The following problems correspond with the four cases of oblique angled triangles; (Art. 148.) but are equally adapted to right angled triangles.

169. PROB. I. *The angles and one side* of a triangle being given; to find, by construction, the other two sides.

Draw the given side. From the ends of it, lay off two of the given angles. Extend the other sides till they intersect; and then measure their lengths, on a scale of equal parts.

Ex. 1. Given the side b 32 rods, (Fig. 27.) the angle A $56^\circ 20'$, and the angle C $49^\circ 10'$; to construct the triangle, and find the lengths of the sides a and c .

From a scale of equal parts, make $b=32$. With the line of chords, make, at one end of b , an angle of $56^\circ 20'$, and at the other end, an angle of $49^\circ 10'$. From A and C , draw the lines a and c , till they meet in B . Their lengths, on the same scale from which b was taken, will be 25 and $27\frac{1}{2}$.

2. In a right angled triangle, (Fig. 17.) given the *hypoth-*

* Sometimes the line of longitude is placed *under* the line of chords.

enuse 90, and the angle $A\ 32^\circ\ 20'$, to find the base and perpendicular.

Draw the hypotenuse, and make the angle A equal to $32^\circ\ 20'$. Extend AB indefinitely, and make the angle C equal to the complement of A ; or let fall a perpendicular from C upon AB . (Euc. 12. 1.) The length of AB will be 76, of BC 48.

3. Given the side AC 68, the angle $A\ 124^\circ$, and the angle $C\ 37^\circ$; to construct the triangle.

170. PROB. II. *Two sides and an opposite angle being given, to find the remaining side, and the other two angles.*

Draw one of the given sides; from one end of it, lay off the given angle; and extend a line indefinitely, for the required side. From the other end of the first side, with the remaining given side for radius, describe an arc cutting the indefinite line. The point of intersection will be the end of the required side.

If the side opposite the given angle be less than the other given side, the case will be *ambiguous*. (Art. 152.)

Ex. 1. Given the angle $A\ 63^\circ\ 35'$ (Fig. 29.) the side $b\ 32$, and the side $a\ 36$.

Draw the side $b\ 32$, make the angle $A\ 63^\circ\ 35'$, and extend AB indefinitely. From C , with 36 for radius, describe an arc cutting the indefinite line in B , and draw the side a from B to C . The side AB will be 36 nearly, the angle $B\ 52^\circ\ 45\frac{1}{2}'$ and $C\ 63^\circ\ 39\frac{1}{2}'$.

2. Given the angle A (Fig. 28.) $35^\circ\ 20'$; the opposite side $a\ 25$, and the side $b\ 35$.

Draw the side $b\ 35$, make the angle $A\ 35^\circ\ 20'$ and extend AH indefinitely. From C , with radius 25 , describe an arc cutting AH in B and B' . Draw CB and CB' , and two triangles will be formed, ABC and $AB'C$, each corresponding with the conditions of the problem.

3. Given the angle $A\ 116^\circ$, the opposite side $a\ 38$, and the side $b\ 26$; to construct the triangle.

171. PROB. III. *Two sides and the included angle being given; to find the other side and angles.*

Draw one of the given sides. From one end of it, lay off the given angle, and draw the other given side. Then connect the extremities of this and the first line.

Ex. 1. Given the angle A (Fig. 30.) $26^\circ\ 14'$, the side $b\ 78$, and the side $c\ 106$; to find B , C , and a .

Draw the side c 106, at A make an angle of $26^{\circ} 14'$, draw b 78, and connect B and C . The side a will be 50, the angle B $43^{\circ} 44'$, and C $110^{\circ} 2'$.

2. Given A 86° , b 65, and c 83; to find B , C , and a .

172. PROB. IV. The *three sides* being given; to find the angles.

Draw one of the sides, and from one end of it, with an extent equal to the second side, describe an arc. From the other end, with an extent equal to the third side, describe a second arc cutting the first; and from the point of intersection, draw the two sides. (Euc. 22. 1.)

Ex. 1. Given AB (Fig. 31.) 78, AC 70, and BC 54; to find the angles.

Draw AB 78, and from A as a centre, with 70 for radius, describe an arc. From B as a centre, with 54 for radius, describe another arc cutting the first in C ; and from C to the extremities of AB , draw CA and CB . The angles will be A $42^{\circ} 22'$, B $60^{\circ} 52'\frac{2}{3}$, and C $76^{\circ} 45'\frac{1}{3}$.

2. Given the three sides 55, 39, and 46; to find the angles.

173. Any right lined figure whatever, whose sides and angles are given, may be constructed, by laying down the sides from a scale of equal parts, and the angles from a line of chords.

Ex. Given the sides AB (Fig. 35.) = 20, BC = 22, CD = 30, DE = 12; and the angles B = 102° , C = 130° , D = 108° , to construct the figure.

Draw the side AB = 20, make the angle B = 102° , draw BC = 22, make C = 130° , draw CD = 30, make D = 108° , draw DE = 12, and connect E and A .

The last line EA may be measured on the scale of equal parts; and the angles E and A , by a line of chords.

SECTION VI.

DESCRIPTION AND USE OF GUNTER'S SCALE.

ART. 174. **A**N expeditious method of solving the problems in trigonometry, and making other logarithmic calculations, in a mechanical way, has been contrived by Mr. Edmund Gunter. The logarithms of numbers, of sines, tangents, &c. are represented by *lines*. By means of these, multiplication, division, the rule of three, involution, evolution, &c. may be performed much more rapidly, than in the usual method by figures.

The logarithmic lines are generally placed on one side only of the scale in common use. They are,

A line of artific'l <i>Sines</i> div'd into <i>Rhumbs</i> , and marked	S. R.
A line of artificial <i>Tangents</i> , do.	T. R.
A line of the logarithms of <i>numbers</i> ,	Num.
A line of artificial <i>Sines</i> , to every <i>degree</i> ,	SIN.
A line of artificial <i>Tangents</i> , do.	TAN.
A line of <i>Versed Sines</i> ,	V. S.

To these are added a line of *equal parts*, and a line of *Meridional Parts*, which are not logarithmic. The latter is used in Navigation.

The Line of Numbers.

175. Portions of the line of *Numbers*, are intended to represent the *logarithms* of the natural series of numbers 2, 3, 4, 5, &c.

The logarithms of 10, 100, 1000, &c. are 1, 2, 3, &c. (Art. 3.)

If then, the log. of 10 be represented by a line of 1 foot;
the log. of 100 will be repres'd by one of 2 feet;
the log. of 1000 by one of 3 feet;
the lengths of the several lines being proportional to the corresponding logarithms in the tables. *Portions* of a foot will represent the logarithms of numbers between 1 and 10;

and portions of a line 2 feet long, the logarithms of numbers between 1 and 100.

On Gunter's scale, the line of the logarithms of numbers begins at a brass pin on the left, and the divisions are numbered 1, 2, 3, &c. to another pin near the middle. From this, the numbers are repeated, 2, 3, 4, &c. which may be read 20, 30, 40, &c. The logarithms of numbers between 1 and 10 are represented by portions of the first half of the line; and the logarithms of numbers between 10 and 100, by portions greater than half the line, and less than the whole.

176. The logarithm of 1, which is 0, is denoted, not by any extent of line, but by a *point* under 1, at the commencement of the scale. The distances from this point to different parts of the line, represent other logarithms, of which the *figures* placed over the several divisions are the *natural numbers*. For the intervening logarithms, the intervals between the figures, are divided into tenths, and sometimes into smaller portions. On the right hand half of the scale, as the divisions which are numbered are *tens*, the subdivisions are units.

Ex. 1. To take from the scale the logarithm of 3.6; set one foot of the compasses under 1 at the beginning of the scale, and extend the other to the 6th division after the first figure 3.

2. For the logarithm of 47; extend from 1 at the beginning, to the 7th subdivision after the second figure 4.*

177. It will be observed, that the divisions and subdivisions *decrease*, from left to right; as in the tables of *logarithms*, the differences decrease. The difference between the logarithms of 10 and 100 is no greater, than the difference between the logarithms of 1 and 10.

178. The line of numbers, as it has been here explained, furnishes the logarithms of all numbers between 1 and 100. And if the *indices* of the logarithms be neglected, the same scale may answer for all numbers whatever. For the *decimal* part of the logarithm of any number is the same, as that of the number multiplied or divided by 10, 100, &c. (Art. 14.) In logarithmic calculations, the use of the indices is to determine the distance of the several figures of the natural numbers from the place of units. (Art. 11.) But in those cases in which the logarithmic line is commonly used,

*If the compasses will not reach the distance required; first open them so as to take off *half*, or any part of the distance, and then the remaining part.

it will not generally be difficult to determine the local value of the figures in the result.

179. We may, therefore, consider the *point* under 1 at the left hand, as representing the logarithm of 1, or 10, or 100; or $\frac{1}{10}$, or $\frac{1}{100}$, &c. for the decimal part of the logarithm of each of these is 0. But if the first 1 is reckoned 10, all the succeeding numbers must also be increased in a tenfold ratio; so as to read, on the first half of the line, 20, 30, 40, &c. and on the other half, 200, 300, &c.

The whole extent of the logarithmic line,
 is from 1 to 100, or from 0.1 to 10,
 or from 10 to 1000, or from 0.01 to 1,
 or from 100 to 10000, &c. or from 0.001 to 0.1, &c.

Different values may, on different occasions, be assigned to the several numbers and subdivisions marked on this line. But for any one calculation, the value must remain the same.

Ex. Take from the scale 365.

As this number is between 10 and 1000, let the 1 at the beginning of the scale, be reckoned 10. Then, from this point to the second 3 is 300; to the 6th dividing stroke is 60; and half way from this to the next stroke is 5.

180. Multiplication, division, &c. are performed by the line of numbers, on the same principle, as by common logarithms. Thus,

To *multiply* by this line, *add* the logarithms of the two factors; (Art. 37.) that is, take off, with the compasses, that length of line which represents the logarithm of *one* of the factors, and apply this so as to extend forward from the end of that which represents the logarithm of the *other* factor. The sum of the two will reach to the end of the line representing the logarithm of the product.

Ex. Multiply 9 into 8. The extent from 1 to 8, added to that from 1 to 9, will be equal to the extent from 1 to 72 the product.

181. To *divide* by the logarithmic line, *subtract* the logarithm of the divisor from that of the dividend; (Art. 41.) that is, take off the logarithm of the divisor, and this extent set back from the end of the logarithm of the dividend, will reach to the logarithm of the quotient.

Ex. Divide 42 by 7. The extent from 1 to 7, set back from 42, will reach to 6 the quotient.

182. *Involution* is performed in logarithms, by multiplying the logarithm of the quantity into the index of the power; (Art. 45.) that is, by *repeating* the logarithm, as many times

as there are units in the index. To involve a quantity on the scale, then, take in the compasses the linear logarithm, and double it, treble it, &c. according to the index of the proposed power.

Ex. 1. Required the square of 9. Extend the compasses from 1 to 9. Twice this extent will reach to 81 the square.

2. Required the cube of 4. The extent from 1 to 4, repeated three times, will reach to 64 the cube of 4.

183. On the other hand, to perform *evolution* on the scale; take half, one third, &c. of the logarithm of the quantity, according to the index of the proposed root.

Ex. 1. Required the square root of 49. Half the extent from 1 to 49, will reach from 1 to 7 the root.

2. Required the cube root of 27. One third the distance from 1 to 27, will extend from 1 to 3 the root.

184. The *Rule of Three* may be performed on the scale, in the same manner as in logarithms, by adding the two middle terms, and from the sum, subtracting the first term. (Art. 52.) But it is more convenient in practice to begin by subtracting the first term from one of the others. If four quantities are proportional, the quotient of the first divided by the second, is equal to the quotient of the third divided by the fourth. (Alg. 364.)

Thus if $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{c} = \frac{b}{d}$. (Alg. 360.)

But in logarithms, subtraction takes the place of division; so that,

$\log. a - \log. b = \log. c - \log. d$. Or $\log. a - \log. c = \log. b - \log. d$.

Hence,

185. On the scale, the difference between the first and second terms of a proportion, is equal to the difference between the third and fourth. Or, the difference between the first and third terms, is equal to the difference between the second and fourth.

The difference between the two terms is taken, by extending the compasses from one to the other. If the second term be greater than the first; the fourth must be greater than the third; if less, less. (Alg. 395.)* Therefore if the compasses extend forward from left to right, that is, from a less number to a greater, from the first term to the second; they must also extend forward from the third to the fourth. But if they extend backward, from the first term to the sec-

* Euc. 14. 5.

ond, they must extend the same way, from the third to the fourth.

Ex. 1. In the proportion $3:8::12:32$, the extent from 3 to 8, will reach from 12 to 32: Or, the extent from 3 to 12, will reach from 8 to 32.

2. If 54 yards of cloth cost 48 dollars, what will 18 yards cost? $54:48::18:16$

The extent from 54 to 48, will reach *backwards* from 18 to 16.

3. If 63 gallons of wine cost 81 dollars, what will 35 gallons cost? $63:81::35:45$

The extent from 63 to 81, will reach from 35 to 45.

The Line of Sines.

186. The line on Gunter's scale marked SIN. is a line of logarithmic sines, made to correspond with the line of numbers. The whole extent of the line of numbers, (Art. 179.) is from 1 to 100, whose logs. are 0.00000 and 2.00000, or from 10 to 1000, whose logs. are 1.00000 and 3.00000, or from 100 to 10000, whose logs. are 2.00000 and 4.00000, the *difference of the indices* of the two extreme logarithms being in each case 2.

Now the logarithmic sine of $0^\circ 34' 22'' 41'''$ is 6.00000

And the sine of 90° (Art. 95.) is 10.00000

Here also the difference of the indices is 2. If then the point directly beneath one extremity of the line of numbers, be marked for the sine of $0^\circ 34' 22'' 41'''$; and the point beneath the other extremity, for the sine of 90° ; the interval may furnish the intermediate sines; the divisions on it being made to correspond with the decimal part of the logarithmic sines in the tables.*

The first dividing stroke in the line of Sines is generally at $0^\circ 40'$, a little farther to the right than the beginning of the line of Numbers. The next division is at $0^\circ 50'$; then begins the numbering of the degrees, 1, 2, 3, 4, &c. from left to right.

*To represent the sines *less* than $34' 22'' 41'''$, the scale must be extended on the left indefinitely. For, as the sine of an arc approaches to 0, its logarithm, which is negative, increases without limit. (Art. 15.)

The line of Tangents.

187. The first 45 degrees on this line are numbered from left to right, nearly in the same manner as on the line of Sines.

The logarithmic tangent of $0^{\circ} 34' 22'' 35'''$ is 8.00000

And the tangent of 45° , (Art. 95.) is 10.00000

The difference of the indices being 2, 45 degrees will reach to the end of the line. For those above 45° , the scale ought to be continued much farther to the right. But, as this would be inconvenient, the numbering of the degrees, after reaching 45, is *carried back* from right to left. The same dividing stroke answers for an arc and its *complement*, one above and the other below 45° . For, (Art. 93. Propor. 9.)

$$\tan : R :: R : \cot$$

In logarithms, therefore, (Art. 184.)

$$\tan - R = R - \cot.$$

That is, the *difference* between the tangent and radius, is equal to the difference between radius and the cotangent: in other words, one is as much *greater* than the tangent of 45° , as the other is *less*. In taking, then, the tangent of an arc greater than 45° , we are to suppose the distance between 45 and the division marked with the given number of degrees, to be *added* to the whole line, in the same manner as if the line were continued out. In working proportions, extending the compasses *back*, from the tangent of 45° , must be considered the same as carrying them *forward* in other cases. See art. 185.

Trigonometrical Proportions on the Scale.

188. In working proportions in trigonometry by the scale; *the extent from the first term to the middle term of the same name, will reach from the other middle term to the fourth term.* (Art. 185.)

In a trigonometrical proportion, two of the terms are the lengths of sides of the given triangle; and the other two are tabular sines, tangents, &c. The former are to be taken from the line of numbers; the latter, from the lines of logarithmic sines and tangents. If one of the terms is a *secant*, the calculation cannot be made on the scale, which has commonly no line of secants. It must be kept in mind that *ra-*

dius is equal to the sine of 90° , or to the tangent of 45° . (Art. 95.) Therefore, whenever radius is a term in the proportion, one foot of the compasses must be set on the end of the line of sines or of tangents.

189. The following examples are taken from the proportions which have already been solved by numerical calculation.

Ex. 1. In Case I of right angled triangles, (Art. 134. ex.1.)

$$R : 45 :: \text{Sin } 32^\circ 20' : 24$$

Here the third term is a *sine*; the first term radius is, therefore, to be considered as the sine of 90° . Then the extent from 90° to $32^\circ 20'$ on the line of sines, will reach from 45 to 24 on the line of numbers. As the compasses are set back from 90° to $32^\circ 20'$; they must also be set back from 45. (Art. 185.)

2. In the same case, if the base be made radius, (Page 60.)

$$R : 38 :: \text{Tan } 32^\circ 20' : 24$$

Here, as the third term is a *tangent*, the first term radius is to be considered the tangent of 45° . Then the extent from 45° to $32^\circ 20'$ on the line of tangents, will reach from 38 to 24 on the line of numbers.

3. If the perpendicular be made radius, (Page 60.)

$$R : 24 :: \text{Tan } 57^\circ 40' : 38$$

The extent from 45° to $57^\circ 40'$ on the line of tangents, will reach from 24 to 38 on the line of numbers. For the tangent of $57^\circ 40'$ on the scale, look for its *complement* $32^\circ 20'$. (Art. 187.) In this example, although the compasses extend back from 45° to $57^\circ 40'$; yet, as this is from a *less* number to a *greater*, they must extend *forward* on the line of numbers. (Art. 185. 187.)

4. In art. 135, $35 : R :: 26 : \text{Sin } 48^\circ$

The extent from 35 to 26 will reach from 90° to 48° .

5. In art. 136, $R : 48 :: \text{Tan } 27^\circ \frac{1}{4} : 24 \frac{3}{4}$

The extent from 45° to $27^\circ \frac{1}{4}$, will reach from 48 to $24 \frac{3}{4}$.

6. In art. 150, ex. 1. $\text{Sin } 74^\circ 30' : 32 :: \text{Sin } 56^\circ 20' : 27 \frac{1}{2}$.

For other examples, see the several cases in Sections III and IV.

190. Though the solutions in trigonometry may be effected, by the logarithmic scale, or by geometrical construction, as well as by arithmetical computation; yet the latter method is by far the most accurate. The first is valuable principally for the *expedition* with which the calculations are made by it. The second is of use, in presenting the *form* of the triangle to the eye. But the accuracy which attends arithmetical operations, is not to be expected, in taking lines from a scale, with a pair of compasses.*

* See note I.

SECTION VII.*

THE FIRST PRINCIPLES OF TRIGONOMETRICAL ANALYSIS.

ART. 191. **I**N the preceding sections, sines, tangents, and secants, have been employed in calculating the sides and angles of triangles. But the use of these lines is not confined to this object. Important assistance is derived from them, in conducting many of the investigations in the higher branches of analysis, particularly in physical astronomy. It does not belong to an elementary treatise of trigonometry, to prosecute these inquiries to any considerable extent. But this is the proper place for *preparing the formulæ*, the applications of which are to be made elsewhere.

Positive and negative signs in trigonometry.

192. Before entering on a particular consideration of the algebraic expressions which are produced by combinations of the several trigonometrical lines, it will be necessary to attend to the positive and negative *signs* in the different quarters of the circle. The sines, tangents, &c. in the tables, are calculated for a single quadrant only. But these are made to answer for the whole circle. For they are of the same length, in each of the four quadrants. (Art. 90.) Some of them, however, are *positive*; while others are *negative*. In algebraic processes, this distinction must not be neglected.

193. For the purpose of tracing the changes of the signs, in different parts of the circle, let it be supposed that a straight line CT (Fig. 36.) is fixed at one end C, while the other end is carried round, like a rod moving on a pivot; so that the point S shall describe the circle ABDH. If the two

*Euler's Analysis of Infinites, Hutton's Mathematics, Lacroix's Differential Calculus, Mansfield's Essays, Legendre's, Lacroix's, Playfair's, and Woodhouse's Trigonometry.

diameters AD and BH be perpendicular to each other, they will divide the circle into quadrants.

194. In the *first quadrant* AB, the sine, cosine, tangent, &c. are considered *all positive*. In the *second quadrant* BD, the sine P'S' continues *positive*; because it is still on the *upper* side of the diameter AD, from which it is measured. But the *cosine*, which is measured from BH, becomes *negative*, as soon as it changes from the *right* to the *left* of this line. (Alg. 507.) In the *third quadrant*, the *sine* becomes *negative*, by changing from the upper side to the under side of DA. The *cosine* continues *negative*, being still on the *left* of BH. In the *fourth quadrant*, the *sine* continues *negative*. But the *cosine* becomes *positive*, by passing to the *right* of BH.

195. The signs of the *tangents* and *secants* may be derived from those of the sines and cosines. The relations of these several lines to each other must be such, that a uniform method of calculation may extend through the different quadrants.

In the first quadrant, (Art. 93. Propor. 1.)

$$R : \cos :: \tan : \sin, \text{ that is, } \tan = \frac{R \times \sin}{\cos}.$$

The sign of the quotient is determined from the signs of the divisor and dividend. (Alg. 123.) The radius is considered as always positive. If then the sine and cosine be both positive, or both negative, the tangent will be positive. But if one of these be positive, while the other is negative, the tangent will be negative.

Now by the preceding article,

In the 2d quadrant, the sine is positive, and the cosine negative.

The tangent must therefore be *negative*.

In the 3d quadrant, the sine and cosine are both negative.

The tangent must therefore be *positive*.

In the ~~fourth~~ ^{fourth} quadrant, the sine is negative, and the cosine positive.

The tangent must therefore be *negative*.

196. By the 9th, 3d, and 6th proportions in art. 93,

$$1. \tan : R :: R : \cot, \text{ that is, } \cot = \frac{R^2}{\tan}.$$

Therefore, as radius is uniformly positive, the *cotangent* must have the same sign as the tangent.

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$$2. \cos : R :: R : \sec, \text{ that is, } \sec = \frac{R^2}{\cos}.$$

The *secant*, therefore, must have the same sign as the *co-sine*.

$$3. \sin : R :: R : \operatorname{cosec}, \text{ that is, } \operatorname{Cosec} = \frac{R^2}{\sin}.$$

The *cosecant*, therefore, must have the same sign as the *sine*.

The *versed sine*, as it is measured from A, in one direction only, is invariably positive.

197. The *tangent* AT (Fig. 36.) increases, as the arc extends from A towards B. See also Fig. 11. Near B the increase is very rapid; and when the difference between the arc and 90° , is less than any assignable quantity, the tangent is greater than any assignable quantity, and is said to be *infinite*. (Alg. 447.) If the arc is *exactly* 90° degrees, it has, strictly speaking, *no tangent*. For a tangent is a line, drawn perpendicular to the diameter which passes through one end of the arc, and extended till it meets a line proceeding from the centre through the other end. (Art. 64.) But if the arc is 90° degrees, as AB (Fig. 36.) the angle ACB is a right angle, and therefore AT is *parallel* to CB; so that, if these lines be extended ever so far, they can never meet. Still, as an arc infinitely near to 90° has a tangent infinitely great, it is frequently said, in concise terms, that the tangent of 90° is infinite.

In the second quadrant, the tangent is, at first, infinitely great, and gradually diminishes, till at D it is reduced to nothing. In the third quadrant it increases again, becomes infinite near H, and is reduced to nothing at A.

The *cotangent* is inversely as the tangent. It is therefore nothing at B and H, (Fig. 36.) and infinite near A and D.

198. The *secant* increases with the tangent, through the first quadrant, and becomes infinite near B; it then diminishes, in the second quadrant, till at D it is equal to the radius CD. In the third quadrant, it increases again, becomes infinite near H, after which it diminishes, till it becomes equal to radius.

The *cosecant* decreases, as the secant increases, and *v. v.* It is therefore equal to radius at B and H, and infinite near A and D.

199. The *sine* increases through the first quadrant, till at

(Fig. 36.) it is equal to radius. See also Fig. 13. It then diminishes, and is reduced to nothing at D. In the third quadrant, it increases again, becomes equal to radius at H, and is reduced to nothing at A.

The *cosine* decreases through the first quadrant, and is reduced to nothing at B. In the second quadrant, it increases and it becomes equal to radius at D. It then diminishes again, is reduced to nothing at H, and afterwards increases and it becomes equal to radius at A.

In all these cases, the arc is supposed to begin at A, and to extend round in the direction of BDH.

200. The *sine* and *cosine* vary from nothing to radius, which they never exceed. The *secant* and *cosecant* are never less than radius, but may be greater than any given length. The *tangent* and *cotangent* have every value from nothing to infinity. Each of these lines, after reaching its *greatest* limit, begins to *decrease*; and as soon as it arrives at its *least* limit, begins to *increase*. Thus the *sine* begins to decrease, after becoming equal to radius, which is its greatest limit. But the *secant* begins to increase after becoming equal to radius, which is its least limit.

201. The substance of several of the preceding articles, comprised in the following tables. The first shows the *signs* of the trigonometrical lines, in each of the quadrants of the circle. The other gives the *values* of these lines, at the extremity of each quadrant.

	Quadrant			
	1st	2d	3d	4th
Sine and cosecant	+	+	—	—
Cosine and secant	+	—	—	+
Tangent and cotangent	+	—	+	—

	0°	90°	180°	270°	360°
Sine	0	r	0	r	0
Cosine	r	0	r	0	r
Tangent	0	∞	0	∞	0
Cotangent	∞	0	∞	0	∞
Secant	r	∞	r	∞	r
Cosecant	∞	r	∞	r	∞

Here r is put for radius, and ∞ for infinite.

202. By comparing these two tables, it will be seen, that each of the trigonometrical lines changes from positive to

negative, or from negative to positive, in that part of the circle in which the line is either *nothing* or *infinite*. Thus the tangent changes from positive to negative, in passing from the first quadrant to the second, through the place where it is infinite. It becomes positive again, in passing from the second quadrant to the third, through the point in which it is nothing.

203. There can be no more than 360 degrees in any circle. But a body may have a number of successive revolutions, in the same circle; as the earth moves round the sun, nearly in the same orbit, year after year. In astronomical calculations, it is frequently necessary to add together parts of different revolutions. The sum may be more than 360°. But a body which has made more than a complete revolution in a circle, is only brought back to a point which it had passed over before. So the sine, tangent, &c. of an arc greater than 360°, is the same as the sine, tangent, &c. of some arc less than 360°. If an entire circumference, or a number of circumferences be added to any arc, it will terminate in the same point as before. So that, if C be put for a whole circumference or 360°, and x be any arc whatever;

$$\begin{aligned}\sin x &= \sin (C+x) = \sin (2C+x) = \sin (3C+x), \text{ \&c.} \\ \tan x &= \tan (C+x) = \tan (2C+x) = \tan (3C+x), \text{ \&c.}\end{aligned}$$

204. It is evident also, that, in a number of successive revolutions, in the same circle;

The first quadrant must coincide with the 5th,	9th, 13th, 17th,
The second, with the	6th, 10th, 14th, 18th, &c.
The third, with the	7th, 11th, 15th, 19th, &c.
The fourth, with the	8th, 12th, 16th, 20th, &c.

205. If an arc extending in a certain direction from a given point, be considered *positive*; an arc extending from the same point, in an *opposite* direction, is to be considered *negative*. (Alg. 507.) Thus, if the arc extending from A to S (Fig. 36.) be positive; an arc extending from A to S''' will be negative. The latter will not terminate in the *same quadrant* as the other; and the signs of the tabular lines must be accommodated to this circumstance. Thus the sine of AS will be positive, while that of AS''' will be negative. (Art. 194.) When a greater arc is subtracted from a less, if the latter be positive, the *remainder* must be negative. (Alg. 58, 9.)

TRIGONOMETRICAL FORMULÆ.

206. From the view which has here been taken of the changes in the trigonometrical lines, it will be easy to see, in what parts of the circle each of them increases or decreases. But this does not determine their exact values, except at the extremities of the several quadrants. In the analytical investigations which are carried on by means of these lines, it is necessary to calculate the changes produced in them, by a given increase or diminution of the arcs to which they belong. In this there would be no difficulty, if the sines, tangents, &c. were *proportioned* to their arcs. But this is far from being the case. If an arc is doubled, its sine is *not* exactly doubled. Neither is its tangent or secant. We have to inquire, then, in what manner, the sine, tangent, &c. of one arc may be obtained, from those of other arcs already known.

The problem on which almost the whole of this branch of analysis depends, consists in deriving, from the sines and cosines of two given arcs, expressions for the sine and cosine of their *sum* and *difference*. For, by addition and subtraction, a few arcs may be so combined and varied, as to produce others of almost every dimension. And the expressions for the tangents and secants may be deduced from those of the sines and cosines.

∴ *Expressions for the SINE and COSINE of the SUM and DIFFERENCE of arcs.*

207. Let $a = AH$, the greater of the given arcs,
And $b = HL = HD$, the less. (Fig. 37.)

Then $a + b = AH + HL = AL$, the *sum* of the two arcs,
And $a - b = AH - HD = AD$, their *difference*.

Draw the chord DL , and the radius CH , which may be represented by R . As DH is, by construction, equal to HL ; DQ is equal to QL , and therefore DL is perpendicular to CH . (Euc. 3. 3.) Draw DO , HN , QP , and LM , each perpendicular to AC ; and DS and QB parallel to AC .

From the definitions of the sine and cosine, (Art. 82, 9,) it is evident, that

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The sine $\left\{ \begin{array}{l} \text{of AH, that is, } \sin a = \text{HN,} \\ \text{of HL, that is, } \sin b = \text{QL,} \\ \text{of AL, i.e. } \sin(a+b) = \text{LM,} \\ \text{of AD, i.e. } \sin(a-b) = \text{DO.} \end{array} \right.$

The cosine $\left\{ \begin{array}{l} \text{of AH, that is, } \cos a = \text{CN,} \\ \text{of HL, that is, } \cos b = \text{CQ,} \\ \text{of AL, i.e. } \cos(a+b) = \text{CM,} \\ \text{of AD, i.e. } \cos(a-b) = \text{CO.} \end{array} \right.$

The triangle CHN is obviously similar to CQP; and it is also similar to BLQ, because the sides of the one are perpendicular to those of the other, each to each. We have, then,

1. CH : CQ :: HN : QP, that is, R : $\cos b$:: $\sin a$: QP
2. CH : QL :: CN : BL, R : $\sin b$:: $\cos a$: BL,
3. CH : CQ :: CN : CP, R : $\cos b$:: $\cos a$: CP,
4. CH : QL :: HN : QB, R : $\sin b$:: $\sin a$: QB,

Converting each of these proportions into an equation;

$$1. \quad QP = \frac{\sin a \cos b^*}{R}$$

$$3. \quad CP = \frac{\cos a \cos b}{R}$$

$$2. \quad BL = \frac{\sin b \cos a}{R}$$

$$4. \quad QB = \frac{\sin a \sin b}{R}$$

Then adding the first and second,

$$QP + BL = \frac{\sin a \cos b + \sin b \cos a}{R}$$

Subtracting the second from the first,

$$QP - BL = \frac{\sin a \cos b - \sin b \cos a}{R}$$

Subtracting the fourth from the third,

$$CP - QB = \frac{\cos a \cos b - \sin a \sin b}{R}$$

Adding the third and fourth,

$$CP + QB = \frac{\cos a \cos b + \sin a \sin b}{R}$$

* In these formulæ, the sign of multiplication is omitted; $\sin a \cos b$ being put for $\sin a \times \cos b$, that is the product of the sine of a into the cosine of b .

But it will be seen, from the figure, that

$$QP + BL = BM + BL = LM = \sin(a+b)$$

$$QP - BL = QP - QS = DO = \sin(a-b)$$

$$CP - QB = CP - PM = CM = \cos(a+b)$$

$$CP + QB = CP + SD = CO = \cos(a-b)$$

206. If then, for the first member of each of the four equations above, we substitute its value, we shall have,

$$I. \sin(a+b) = \frac{\sin a \cos b + \sin b \cos a}{R}$$

$$II. \sin(a-b) = \frac{\sin a \cos b - \sin b \cos a}{R}$$

$$III. \cos(a+b) = \frac{\cos a \cos b - \sin a \sin b}{R}$$

$$IV. \cos(a-b) = \frac{\cos a \cos b + \sin a \sin b}{R}$$

Or, multiplying both sides by R ,

$$R \sin(a+b) = \sin a \cos b + \sin b \cos a,$$

$$R \sin(a-b) = \sin a \cos b - \sin b \cos a$$

$$R \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$R \cos(a-b) = \cos a \cos b + \sin a \sin b$$

That is, the product of radius and the *sine* of the *sum* of two arcs, is equal to the product of the *sine* of the first arc into the *cosine* of the second + the product of the *sine* of the second into the *cosine* of the first.

The product of radius and the *sine* of the *difference* of two arcs, is equal to the product of the *sine* of the first arc into the *cosine* of the second - the product of the *sine* of the second into the *cosine* of the first.

The product of radius and the *cosine* of the *sum* of two arcs, is equal to the product of the *cosines* of the arcs - the product of their *sines*.

The product of radius and the *cosine* of the *difference* of two arcs, is equal to the product of the *cosines* of the arcs + the product of their *sines*.

These four equations may be considered as fundamental propositions, in what is called the *Arithmetic of Sines and Cosines*, or *Trigonometrical Analysis*.

Expressions for the sine and cosine of a DOUBLE arc.

209. When the sine and cosine of any arc are given, it is easy to derive from the equations in the preceding article, expressions for the sine and cosine of *double* that arc. As the two arcs a and b may be of any dimensions, they may be supposed to be *equal*. Substituting, then, a for its equal b , the first and the third of the four preceding equations will become,

$$R \sin(a+a) = \sin a \cos a + \sin a \cos a$$

$$R \cos(a+a) = \cos a \cos a - \sin a \sin a$$

That is, writing $\sin^2 a$ for the square of the sine of a , and $\cos^2 a$ for the square of the cosine of a ,

$$\text{I. } R \sin 2a = 2 \sin a \cos a$$

$$\text{II. } R \cos 2a = \cos^2 a - \sin^2 a.$$

Expressions for the sine and cosine of HALF a given arc.

210. The arc in the preceding equations, not being necessarily limited to any particular value, may be *half* a , as well as a . Substituting then $\frac{1}{2}a$ for a , we have,

$$R \sin a = 2 \sin \frac{1}{2}a \cos \frac{1}{2}a$$

$$R \cos a = \cos^2 \frac{1}{2}a - \sin^2 \frac{1}{2}a$$

Putting the sum of the squares of the sine and cosine equal to the square of radius, (Art. 94.) and inverting the members of the last equation,

$$\cos^2 \frac{1}{2}a + \sin^2 \frac{1}{2}a = R^2$$

$$\cos^2 \frac{1}{2}a - \sin^2 \frac{1}{2}a = R \cos a$$

If we *subtract* one of these from the other, the terms containing $\cos^2 \frac{1}{2}a$ will disappear; and if we *add* them, the terms containing $\sin^2 \frac{1}{2}a$ will disappear: therefore,

$$2 \sin^2 \frac{1}{2}a = R^2 - R \cos a$$

$$2 \cos^2 \frac{1}{2}a = R^2 + R \cos a$$

Dividing by 2, and extracting the root of both sides,

$$\text{I. } \sin \frac{1}{2}a = \sqrt{\frac{1}{2}R^2 - \frac{1}{2}R \cos a}$$

$$\text{II. } \cos \frac{1}{2}a = \sqrt{\frac{1}{2}R^2 + \frac{1}{2}R \cos a}$$

Expressions for the sines and cosines of MULTIPLE arcs.

211. In the same manner, as expressions for the sine and

cosine of a *double* arc, are derived from the equations in art. 208; expressions for the sines and cosines of other multiple arcs may be obtained, by substituting successively $2a$, $3a$, &c. for b , or for b and a both. Thus,

$$\text{I. } \begin{cases} R \sin 3a = R \sin(a+2a) = \sin a \cos 2a + \sin 2a \cos a \\ R \sin 4a = R \sin(a+3a) = \sin a \cos 3a + \sin 3a \cos a \\ R \sin 5a = R \sin(a+4a) = \sin a \cos 4a + \sin 4a \cos a \\ \text{\&c.} \end{cases}$$

$$\text{II. } \begin{cases} R \cos 3a = R \cos(a+2a) = \cos a \cos 2a - \sin a \sin 2a \\ R \cos 4a = R \cos(a+3a) = \cos a \cos 3a - \sin a \sin 3a \\ R \cos 5a = R \cos(a+4a) = \cos a \cos 4a - \sin a \sin 4a \\ \text{\&c.} \end{cases}$$

Expressions for the PRODUCTS of sines and cosines.

212. Expressions for the products of sines and cosines may be obtained, by adding and subtracting the four equations in art. 208, viz.

$$\begin{aligned} R \sin(a+b) &= \sin a \cos b + \sin b \cos a \\ R \sin(a-b) &= \sin a \cos b - \sin b \cos a \\ R \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ R \cos(a-b) &= \cos a \cos b + \sin a \sin b \end{aligned}$$

Adding the first and second,

$$R \sin(a+b) + R \sin(a-b) = 2 \sin a \cos b$$

Subtracting the second from the first,

$$R \sin(a+b) - R \sin(a-b) = 2 \sin b \cos a$$

Adding the third and fourth,

$$R \cos(a-b) + R \cos(a+b) = 2 \cos a \cos b$$

Subtracting the third from the fourth,

$$R \cos(a-b) - R \cos(a+b) = 2 \sin a \sin b$$

Inverting the members of each of these equations, and dividing by 2, we have,

$$\begin{aligned} \text{I. } \sin a \cos b &= \frac{1}{2} R \sin(a+b) + \frac{1}{2} R \sin(a-b) \\ \text{II. } \sin b \cos a &= \frac{1}{2} R \sin(a+b) - \frac{1}{2} R \sin(a-b) \\ \text{III. } \cos a \cos b &= \frac{1}{2} R \cos(a-b) + \frac{1}{2} R \cos(a+b) \\ \text{IV. } \sin a \sin b &= \frac{1}{2} R \cos(a-b) - \frac{1}{2} R \cos(a+b) \end{aligned}$$

213. If b be taken *equal* to a , then $a+b=2a$, and $a-b=0$,

the sine of which is 0; (Art. 201.) and the term in which this is a *factor*, is reduced to 0. (Alg. 112.) But the *cosine* of 0 is equal to radius, so that $R \times \cos 0 = R^2$. Reducing, then, the preceding equations,

$$\begin{array}{ll} \text{The first becomes} & \sin a \cos a = \frac{1}{2}R \sin 2a \\ \text{The third,} & \cos^2 a = \frac{1}{2}R^2 + \frac{1}{2}R \cos 2a \\ \text{The fourth,} & \sin^2 a = \frac{1}{2}R^2 - \frac{1}{2}R \cos 2a \end{array}$$

214. If s be the *sum*, and d the *difference* of two arcs, $\frac{1}{2}(s+d)$ will be equal to the greater, and $\frac{1}{2}(s-d)$ to the less. (Art. 153.) Substituting then, in the four equations in art. 212,

$$\begin{array}{ll} s \text{ for } a+b, & \frac{1}{2}(s+d) \text{ for } a, \\ d \text{ for } a-b, & \frac{1}{2}(s-d) \text{ for } b, \end{array} \text{ we have,}$$

$$\begin{array}{ll} \text{I.} & \sin \frac{1}{2}(s+d) \cos \frac{1}{2}(s-d) = \frac{1}{2}R (\sin s + \sin d) \\ \text{II.} & \sin \frac{1}{2}(s-d) \cos \frac{1}{2}(s+d) = \frac{1}{2}R (\sin s - \sin d) \\ \text{III.} & \cos \frac{1}{2}(s+d) \cos \frac{1}{2}(s-d) = \frac{1}{2}R (\cos d + \cos s) \\ \text{IV.} & \sin \frac{1}{2}(s+d) \sin \frac{1}{2}(s-d) = \frac{1}{2}R (\cos d - \cos s) \end{array}$$

215. If radius be taken equal to 1, the two first equations in art. 208, are

$$\begin{array}{l} \sin(a+b) = \sin a \cos b + \sin b \cos a \\ \sin(a-b) = \sin a \cos b - \sin b \cos a \end{array}$$

Multiplying these into each other,

$$\sin(a+b) \times \sin(a-b) = \sin^2 a \cos^2 b - \sin^2 b \cos^2 a$$

But by art. 94, if radius is 1,

$$\cos^2 b = 1 - \sin^2 b, \text{ and } \cos^2 a = 1 - \sin^2 a$$

Substituting, then, for $\cos^2 b$ and $\cos^2 a$, their values, multiplying the factors, and reducing the terms, we have,

$$\sin(a+b) \times \sin(a-b) = \sin^2 a - \sin^2 b$$

Or, because the difference of the squares of two quantities is equal to the product of their sum and difference, [Alg. 235.]

$$\sin(a+b) \times \sin(a-b) = (\sin a + \sin b) \times (\sin a - \sin b)$$

That is, the product of the sine of the sum of two arcs, into the sine of their difference; is equal to the product of the sum of their sines, into the difference of their sines.

Expressions for the TANGENTS of arcs.

216. Expressions for the *tangents* of arcs may be derived from those already obtained for the sines and cosines. By art. 93, proportion 1st,

$$R : \tan :: \cos : \sin$$

$$\text{That is, } \frac{R}{\tan} = \frac{\cos}{\sin}, \text{ and } \frac{\tan}{R} = \frac{\sin}{\cos}, \text{ and } \tan = \frac{R \times \sin}{\cos}.$$

$$\text{Thus } \tan(a+b) = \frac{R \sin(a+b)}{\cos(a+b)}.$$

If, for $\sin(a+b)$ and $\cos(a+b)$ we substitute their values, as given in art. 208, we shall have,

$$\tan(a+b) = \frac{R(\sin a \cos b + \sin b \cos a)}{\cos a \cos b - \sin a \sin b}$$

217. Here, the value of the tangent of the sum of two arcs is expressed, in terms of the *sines* and *cosines* of the arcs. To exchange these for terms of the *tangents*, let the numerator and denominator of the second member of the equation be both divided by $\cos a \cos b$. This will not alter the value of the fraction. (Alg. 140.)

The *numerator*, divided by $\cos a \cos b$, is

$$\frac{R(\sin a \cos b + \sin b \cos a)}{\cos a \cos b} = R \left(\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b} \right) = \tan a + \tan b.$$

And the *denominator*, divided by $\cos a \cos b$, is

$$\frac{\cos a \cos b - \sin a \sin b}{\cos a \cos b} = 1 - \frac{\sin a}{\cos a} \times \frac{\sin b}{\cos b} = 1 - \frac{\tan a}{R} \times \frac{\tan b}{R}$$

$$\text{Therefore } \tan(a+b) = \frac{\tan a + \tan b}{1 - \frac{\tan a \tan b}{R^2}}$$

The denominator of the fraction may be cleared of the divisor R^2 , by multiplying both the numerator and denominator into R^2 . And if we proceed in a similar manner, to find the tangent of $a-b$, we shall have,

$$216. \text{ I. } \tan(a+b) = \frac{R^2(\tan a + \tan b)}{R^2 - \tan a \tan b}$$

$$\text{II. } \tan(a-b) = \frac{R^2(\tan a - \tan b)}{R^2 + \tan a \tan b}$$

If the arcs a and b are equal, then substituting a , $2a$, $3a$, &c. for b , as in art. 211,

$$\tan 2a = \tan(a+a) = \frac{R^2(2\tan a)}{R^2 - \tan^2 a}$$

$$\tan 3a = \tan(a+2a) = \frac{R^2(\tan a + \tan 2a)}{R^2 - \tan a \tan 2a}, \text{ \&c.}$$

219. If we divide the first of the equations in art. 214, by the second; we shall have, after rejecting $\frac{1}{2}R^2$ from the numerator and denominator, (Alg. 140.)

$$\frac{\sin \frac{1}{2}(s+d) \cos \frac{1}{2}(s-d)}{\sin \frac{1}{2}(s-d) \cos \frac{1}{2}(s+d)} = \frac{\sin s + \sin d}{\sin s - \sin d}$$

But the first member of this equation, (Alg. 155,) is equal to

$$\frac{\sin \frac{1}{2}(s+d)}{\cos \frac{1}{2}(s+d)} \times \frac{\cos \frac{1}{2}(s-d)}{\sin \frac{1}{2}(s-d)} = \frac{\tan \frac{1}{2}(s+d)}{R} \times \frac{R}{\tan \frac{1}{2}(s-d)} \quad (\text{Art. 216.})$$

Therefore,

$$\frac{\sin s + \sin d}{\sin s - \sin d} = \frac{\tan \frac{1}{2}(s+d)}{\tan \frac{1}{2}(s-d)}$$

220. According to the notation in art. 214, s stands for the sum of two arcs, and d for their difference. But it is evident that arcs may be taken, whose sum shall be equal to any arc a , and whose difference shall be equal to any arc b , provided that a be greater than b . Substituting then, in the preceding equation, a for s and b for d ,

$$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}. \quad \text{Or,}$$

$$\sin a + \sin b : \sin a - \sin b :: \tan \frac{1}{2}(a+b) : \tan \frac{1}{2}(a-b.)$$

That is, the sum of the sines of two arcs or angles, is to the difference of those sines; as the tangent of half the sum of the arcs or angles, to the tangent of half their difference.

By art. 143, the sides of triangles are as the sines of their opposite angles. It follows, therefore, from the preceding

proposition, (Alg. 389.) that the sum of any two sides of a triangle, is to their difference; as the tangent of half the sum of the opposite angles, to the tangent of half their difference.

This is the second theorem applied to the solution of oblique angled triangles, which was *geometrically* demonstrated in art. 144.*

Expression for the *area* of a triangle, in terms of the sides.

221. Let the sides of the triangle ABC (Fig. 23.) be expressed by a , b , and c , the perpendicular CD by p , the segment AD by d , and the area by S.

Then $a^2 = b^2 + c^2 - 2cd$, (Euc. 13. 2.)

Transposing and dividing by $2c$

$$d = \frac{b^2 + c^2 - a^2}{2c}. \quad \text{Therefore } d^2 = \frac{(b^2 + c^2 - a^2)^2}{4c^2}. \quad \left[\begin{array}{l} \text{Alg.} \\ 223. \end{array} \right]$$

$$\text{By Euc. 47. 1, } p^2 = b^2 - d^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2}$$

Reducing the fraction, (Alg. 150.) and extracting the root of both sides,

$$p = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c} \quad \dagger$$

* See note K.

† The expression for the perpendicular is the same, when one of the angles is *obtuse*, as in Fig. 24. Let AD = d .

Then $a^2 = b^2 + c^2 + 2cd$. (Euc. 12. 2.) And $d = \frac{-b^2 - c^2 + a^2}{2c}$

$$\text{Therefore } d^2 = \frac{(-b^2 - c^2 + a^2)^2}{4c^2} = \frac{(b^2 + c^2 - a^2)^2}{4c^2} \quad (\text{Al. 219.})$$

$$\text{And } p = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c} \quad \text{as above.}$$

O

This gives the length of the *perpendicular*, in terms of the sides of the triangle. But the *area* is equal to the product of the base into half the perpendicular height. (Alg. 518.) that is,

$$S = \frac{1}{2}cp = \frac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}$$

Here we have an expression for the area, in terms of the sides. But this may be reduced to a form much better adapted to arithmetical computation. It will be seen, that the quantities $4b^2c^2$, and $(b^2 + c^2 - a^2)^2$ are both *squares*; and that the whole expression under the radical sign is the *difference* of these squares. But the difference of two squares is equal to the product of the sum and difference of their roots. (Alg. 235.) Therefore $4b^2c^2 - (b^2 + c^2 - a^2)^2$ may be resolved into the two factors,

$$\begin{cases} 2bc + (b^2 + c^2 - a^2) \text{ which is equal to } (b+c)^2 - a^2 \\ 2bc - (b^2 + c^2 - a^2) \text{ which is equal to } a^2 - (b-c)^2 \end{cases}$$

Each of these also, as will be seen in the expressions on the right, is the difference of two squares; and may, on the same principle, be resolved into factors, so that,

$$\begin{cases} (b+c)^2 - a^2 = (b+c+a) \times (b+c-a) \\ a^2 - (b-c)^2 = (a+b-c) \times (a-b+c) \end{cases}$$

Substituting, then, these four factors, in the place of the quantity which has been resolved into them, we have,

$$S = \frac{1}{4}\sqrt{(b+c+a) \times (b+c-a) \times (a+b-c) \times (a-b+c)}$$

Here it will be observed, that all the three sides, a , b , and c , are in each of these factors.

Let $h = \frac{1}{2}(a+b+c)$ half the sum of the sides. Then

$$\begin{aligned} a+b+c &= 2h, \\ b+c-a &= (a+b+c) - 2a = 2h - 2a, \\ a+b-c &= (a+b+c) - 2c = 2h - 2c, \\ a-b+c &= (a+b+c) - 2b = 2h - 2b. \end{aligned}$$

Substituting these values of the several factors, and observing that $2h - 2a = 2(h-a)$ &c.

$$S = \frac{1}{4}\sqrt{2h \times 2(h-a) \times 2(h-b) \times 2(h-c)}$$

The *numeral co-efficients*, here, may be brought together by multiplication. (Alg. 104.) Their product is 16, whose root may be removed from under the radical sign. (Alg. 271.) Therefore,

$$S = \sqrt{h \times (h-a) \times (h-b) \times (h-c)}$$

222. For finding the area of a triangle, then, when the three sides are given, we have this general rule ;

From half the sum of the sides, subtract each side severally ; multiply together the half sum and the three remainders ; and extract the square root of the product.

SECTION VIII.

COMPUTATION OF THE CANON.

ART. 223. **T**HE trigonometrical canon is a set of tables containing the sines, cosines, tangents, &c. to every degree and minute of the quadrant. In the computation of these tables, it is common to find, in the first place, the sine and cosine of *one minute*; and then, by successive additions and multiplications, the sines, cosines, &c. of the larger arcs. For this purpose, it will be proper to begin with an arc whose sine or cosine is a known portion of the radius. The cosine of 60° is equal to *half radius*. (Art. 96. Cor.) A formula has been given, (Art. 210,) by which, when the cosine of an arc is known, the cosine of *half* that arc may be obtained.

By successive bisections of 60° , we have the arcs

30°	$0^\circ\ 28'\ 7''\ 30'''$
15°	$0\ 14\ 3\ 45$
$7^\circ\ 30'$	$0\ 7\ 1\ 52\ 30$
$3^\circ\ 45'$	$0\ 3\ 30\ 56\ 15$
$1^\circ\ 52'\ 30''$	$0\ 1\ 45\ 28\ 7\ 30$
$0^\circ\ 56'\ 15''$	$0^\circ\ 0'\ 52''\ 44''' 3'''' 45'''''$

By formula II, art. 210,

$$\cos \frac{1}{2}a = \sqrt{\frac{1}{2}R^2 + \frac{1}{2}R \times \cos a}$$

If the radius be 1, and if $a=60^\circ$, $b=30^\circ$, $c=15^\circ$, &c. then

$$\cos b = \cos \frac{1}{2}a = \sqrt{\frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} = 0.8660254$$

$$\cos c = \cos \frac{1}{2}b = \sqrt{\frac{1}{2} + \frac{1}{2} \cos b} = 0.9659258$$

$$\cos d = \cos \frac{1}{2}c = \sqrt{\frac{1}{2} + \frac{1}{2} \cos c} = 0.9914449$$

$$\cos e = \cos \frac{1}{2}d = \sqrt{\frac{1}{2} + \frac{1}{2} \cos d} = 0.9978589$$

Proceeding in this manner, by repeated extractions of the square root, we shall find the cosine of

COMPUTATION OF THE CANON. 109

$0^{\circ} 0' 52'' 44''' 3''' 45''''$ to be 0.99999996732

And the sine (Art. 94.) = $\sqrt{1 - \cos^2} = 0.00025566346$

This, however, does not give the sine of *one minute* exactly. The arc is a little *less* than a minute. But the ratio of very small arcs to each other, is so nearly equal to the ratio of their sines, that one may be taken for the other, without sensible error. Now the circumference of a circle is divided into 21600 parts, for the arc of 1'; and into 24576, for the arc of $0^{\circ} 0' 52'' 44''' 3''' 45''''$ Therefore

$$21600 : 24576 :: 0.00025566346 : 0.0002908882.$$

which is the sine of 1 minute very nearly.*

And the cosine = $\sqrt{1 - \sin^2} = 0.9999999577$.

224. Having computed the sine and cosine of one minute, we may proceed, in a contrary order, to find the sines and cosines of *larger* arcs.

Making radius = 1, and adding the two first equations in art. 208, we have

$$\sin(a+b) + \sin(a-b) = 2\sin a \cos b$$

Adding the third and fourth,

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$$

Transposing $\sin(a-b)$ and $\cos(a-b)$

$$\text{I. } \sin(a+b) = 2\sin a \cos b - \sin(a-b)$$

$$\text{II. } \cos(a+b) = 2\cos a \cos b - \cos(a-b)$$

If we put $b=1'$, and $a=1', 2', 3', \&c.$ successively, we shall have expressions for the sines and cosines of a series of arcs increasing regularly by one minute. Thus,

$$\sin(1'+1') = 2\sin 1' \times \cos 1' - \sin 0 = 0.0005817764,$$

$$\sin(2'+1') = 2\sin 2' \times \cos 1' - \sin 1' = 0.0008726645,$$

$$\sin(3'+1') = 2\sin 3' \times \cos 1' - \sin 2' = 0.0011635526,$$

$\&c.$

$\&c.$

$$\cos(1'+1') = 2\cos 1' \times \cos 1' - \cos 0 = 0.9999998308$$

$$\cos(2'+1') = 2\cos 2' \times \cos 1' - \cos 1' = 0.9999996192$$

$$\cos(3'+1') = 2\cos 3' \times \cos 1' - \cos 2' = 0.9999993230$$

$\&c.$

$\&c.$

The constant multiplier here, $\cos 1'$ is 0.9999999577, which is equal to $1 - 0.0000000423$.

* See note L.

225. Calculating, in this manner, the sines and cosines from 1 minute up to 30 degrees, we shall have also the sines and cosines from 60° to 90°. For the sines of arcs between 0° and 30°, are the *cosines* of arcs between 60° and 90°. And the cosines of arcs between 0° and 30°, are the *sines* of arcs between 60° and 90°. (Art. 104.)

226. For the interval between 30° and 60°, the sines and cosines may be obtained by subtraction merely. As twice the sine of 30° is equal to radius; (Art. 96.) by making $a=30^\circ$, the equation marked I, in article 224 will become

$$\sin(30^\circ + b) = \cos b - \sin(30^\circ - b)$$

And putting $b=1', 2', 3', \&c.$ successively,

$$\sin(30^\circ 1') = \cos 1' - \sin(29^\circ 59')$$

$$(30^\circ 2') = \cos 2' - \sin(29^\circ 58')$$

$$(30^\circ 3') = \cos 3' - \sin(29^\circ 57')$$

&c.

&c.

If the *sines* be calculated from 30° to 60°, the *cosines* will also be obtained. For the sines of arcs between 30° and 45°, are the cosines of arcs between 45° and 60°. And the sines of arcs between 45° and 60°, are the cosines of arcs between 30° and 45°.* (Art. 96.)

227. By the methods which have here been explained, the *natural* sines and cosines are found.

The *logarithms* of these, 10 being in each instance added to the index, will be the *artificial* sines and cosines, by which trigonometrical calculations are commonly made. (Art. 102, 3.)

228. The *tangents*, *cotangents*, *secants*, and *cosecants*, are easily derived from the sines and cosines. By art. 93,

$$R : \cos :: \tan : \sin$$

$$\cos : R :: R : \sec$$

$$R : \sin :: \cot : \cos$$

$$\sin : R :: R : \csc$$

Therefore,

$$\text{The tangent} = \frac{R \times \sin}{\cos}$$

$$\text{The secant} = \frac{R^2}{\cos}$$

$$\text{The cotangent} = \frac{R \times \cos}{\sin}$$

$$\text{The cosecant} = \frac{R^2}{\sin}$$

Or if the computations are made by *logarithms*,

$$\text{The tangent} = 10 + \sin - \cos,$$

$$\text{The secant} = 20 - \cos,$$

$$\text{The cotangent} = 10 + \cos - \sin,$$

$$\text{The cosecant} = 20 - \sin.$$

* See note M.

NOTES.

NOTE A. Page 1.

THE name Logarithm is from *λογος* ratio, and *ἀριθμός* number. Considering the ratio of a to 1 as a simple ratio, that of a^2 to 1 is a *duplicate* ratio, of a^3 to 1 a *triplicate* ratio, &c. (Alg. 354.) Here the *exponents* or *logarithms* 2, 3, 4, &c. show how many times the simple ratio is *repeated as a factor*, to form the compound ratio. Thus the ratio of 100 to 1, is the *square* of the ratio of 10 to 1; the ratio of 1000 to 1, is the *cube* of the ratio of 10 to 1, &c. On this account, logarithms are called the *measures* of ratios; that is of the ratios which different numbers bear to unity. See the Introduction to Hutton's Tables, and Mercator's Logarithmo-Technia, in Maseres' Scriptores Logarithmici.

NOTE B. p. 4.

If 1 be added to $-.09691$, it becomes $1-.09691$, which is equal to $+.90309$. The decimal is here rendered positive, by *subtracting* the figures from 1. But it is made 1 too great. This is compensated, by adding -1 to the *integral* part of the logarithm. So that $-2-.09691 = -3+.90309$.

In the same manner, the decimal part of any logarithm which is wholly negative, may be rendered positive, by subtracting it from 1, and adding -1 to the index. The subtraction is most easily performed, by taking the right hand significant figure from 10, and each of the other figures from 9. (Art. 55.)

On the other hand, if the index of a logarithm be negative, while the decimal part is positive; the whole may be rendered negative, by subtracting the decimal part from 1, and taking -1 from the index.

NOTE C. p. 8.

It is common to *define* logarithms to be a series of numbers in arithmetical progression, corresponding with another series in geometrical progression. This is calculated to perplex the learner, when, upon opening the tables, he finds that the natural numbers, as they stand there, instead of being in *geometrical*, are in *arithmetical* progression; and that the logarithms are *not* in arithmetical progression.

It is true, that a geometrical series may be obtained, by taking out, here and there, a few of the natural numbers; and that the logarithms of these will form an arithmetical series. But the definition is not applicable to the whole of the numbers and logarithms, as they stand in the tables.

The supposition that positive and negative numbers have the same series of logarithms, (p. 7.) is attended with some theoretical difficulties. But these do not affect the practical rules for calculating by logarithms.

NOTE D. p. 38.

According to the scheme lately introduced into France, of dividing the denominations of weights, measures, &c. into tenths, hundredths, &c. the fourth part of a circle is divided into 100 degrees, a degree into 100 minutes, a minute into 100 seconds, &c. The whole circle contains 400 of these degrees; a plane triangle 200. If a right angle be taken for the measuring *unit*; degrees, minutes, and seconds, may be written as decimal fractions. Thus $36^{\circ} 5' 49''$ is 0.360549.

According to the French division $\left\{ \begin{array}{l} 10^{\circ} = 9^{\circ} \\ 100' = 54' \\ 1000'' = 324'' \end{array} \right\}$ English.

NOTE E. p. 44.

In Fig. 6th, let the arc $AD = a$, and $ADB = 2a$. Draw BF perpendicular to AH . This will divide the right angled triangle ABH into two similar triangles. (Euc. 8. 6.) The angles ACD and AHB are equal. (Euc. 20. 3.) Therefore

the four triangles ACG, AHB, FHB, and FAB are similar; and the line BH is twice CG, because $BH : CG :: HA : CA$.

The sides of the four triangles are,

$$\begin{array}{lll} AG = \sin a, & CG = \cos a, & HF = \text{vers. sup. } 2a, \\ AB = 2\sin a, & BH = 2\cos a, & AC = \text{the radius,} \\ BF = \sin 2a, & AF = \text{vers } 2a, & AH = \text{the diameter.} \end{array}$$

A variety of proportions may be stated, between the homologous sides of these triangles: For instance,

By comparing the triangles ACG and ABF,

$$\begin{array}{l} AC : AG :: AB : AF, \text{ that is, } R : \sin a :: 2\sin a : \text{vers } 2a \\ AC : CG :: AB : BF \quad R : \cos a :: 2\sin a : \sin 2a \\ AG : CG :: AF : BF \quad \sin a : \cos a :: \text{vers } 2a : \sin 2a \end{array}$$

Therefore,

$$\begin{array}{l} R \times \text{vers } 2a = 2\sin^2 a \\ R \times \sin 2a = 2\sin a \times \cos a \\ \sin a \times \sin 2a = \text{vers } 2a \times \cos a \end{array}$$

By comparing the triangles ACG and BFH,

$$\begin{array}{l} AC : CG :: BH : HF, \text{ that is, } R : \cos a :: 2\cos a : \text{vers. sup. } 2a \\ AG : CG :: BF : HF \quad \sin a : \cos a :: \sin 2a : \text{vers. sup. } 2a \end{array}$$

Therefore,

$$\begin{array}{l} R \times \text{vers. sup. } 2a = 2\cos^2 a \\ \sin a \times \text{vers. sup. } 2a = \cos a \times \sin 2a \\ \&c. \qquad \qquad \&c. \end{array}$$

That is, the product of radius into the versed sine of the supplement of twice a given arc, is equal to twice the square of the cosine of the arc.

And the product of the sine of an arc, into the versed sine of the supplement of twice the arc, is equal to the product of the cosine of the arc, into the sine of twice the arc, &c. &c.

NOTE F. p. 64.

The different methods of solving the same triangle, by making the different sides radius, may serve to verify each other. But they are not, in every instance, equally accurate. The differences in the *sines* of angles near 90° , and in the *cosines* of angles near 0° , are so small, as to leave an uncertainty of several seconds in the result. Thus 9.9999998 is

given in Taylor's tables, as the cosine of every angle between $2' 52''$, and $3' 41''$ and the sine of every angle between $89^{\circ} 56' 19''$ and $89^{\circ} 57' 8''$. But the differences in the *tangents* and *cotangents* are considerable, in every part of the quadrant. The method of solution should be varied, so as to avoid finding a very small angle by its cosine, or one near 90° by its sine. See the latter part of note H.

NOTE G. p. 69.

If the perpendicular be drawn from the angle opposite the longest side, it will always fall *within* the triangle; because the other two angles must, of course, be acute. But if one of the angles at the base be *obtuse*, the perpendicular will fall *without* the triangle, as CP, (Fig. 36.)

In this case, the side on which the perpendicular falls, is to the sum of the other two; as the difference of the latter, to the *sum* of the segments made by the perpendicular.

The demonstration is the same, as in the other case, except that $AH = BP + PA$, instead of $BP - PA$.

Thus in the circle BDHL (Fig. 38.) of which C is the centre,

$$AB \times AH = AL \times AD; \text{ therefore } AB : AD :: AL : AH.$$

$$\text{But } AD = CD + CA = CB + CA$$

$$\text{And } AL = CL - CA = CB - CA$$

$$\text{And } AH = HP + PA = BP + PA$$

Therefore

$$AB : CB + CA :: CB - CA : BP + PA.$$

When the three sides are given, it may be known whether one of the angles is obtuse. For any angle of a triangle is obtuse or acute, according as the square of the side subtending the angle is *greater*, or *less*, than a right angle. (Euc. 12, 13. 2.)

NOTE H. p. 76.

SOLUTIONS OF TRIANGLES.

Any triangle whatever may be solved, by the theorems in section IV. But there are other methods, by which, in cer-

tain circumstances, the calculations are rendered more expeditious, or more accurate results are obtained; particularly, in the *third* and *fourth* cases, in the first of which, two sides and the included angle are given, and in the last, the three sides. See Simson's *Trigonometry*, annexed to his edition of *Euclid*, Legendres' *Trigonometry*, Dr. Maskelyne's Preface to Taylor's *Logarithms*, and Woodhouse's *Trigonometry*.

CASE III.

In astronomical calculations, it is frequently the case, that two sides of a triangle are given by their *logarithms*. By the following proposition, the necessity of finding the corresponding natural numbers is avoided.

THEOREM A. *In any plane triangle, of the two sides which include a given angle, the less is to the greater; as radius to the tangent of an angle greater than 45° :*

And radius is to the tangent of the excess of this angle above 45° ; as the tangent of half the sum of the opposite angles, to the tangent of half their difference.

In the triangle ABC, (Fig. 39.) let the sides AC and AB, and the angle A be given. Through A draw DH perpendicular to AC. Make AD and AF each equal to AC, and AH equal to AB. And let HG be perpendicular to a line drawn from C through F.

Then $AC : AB :: R : \tan ACH$.

And $R : \tan(ACH - 45^\circ) :: \tan \frac{1}{2}(ACB + B) : \tan \frac{1}{2}(ACB - B)$

Demonstration.

In the right angled triangle ACD, as the acute angles are subtended by the equal sides AC and AD, each is 45° . For the same reason, the acute angles in the triangle CAF are each 45° . Therefore, the angle DCF is a right angle, the angles GFH and GHF are each 45° , and the line GH is equal to GF and parallel to DC.

In the triangle ACH, if AC be radius, AH which is equal to AB will be the tangent of ACH. Therefore,

$AC : AB :: R : \tan ACH$.

In the triangle CGH, if CG be radius, GH which is equal to FG will be the tangent of HCG. Therefore,

$$R : \tan (ACH - 45^\circ) :: CG : FG.$$

And, as GH and DC are parallel, (Euc. 2. 6.)

$$CG : FG :: DH : FH.$$

But DH is, by construction, equal to the *sum*, and FH to the *difference* of AC and AB. And by theorem II, [Art. 144.] the sum of the sides is to their difference; as the tangent of half the sum of the opposite angles, to the tangent of half their difference. Therefore,

$$R : \tan (ACH - 45^\circ) :: \tan \frac{1}{2}(ACB + B) : \tan \frac{1}{2}(ACB - B)$$

Ex. In the triangle ABC, (Fig. 30,) given the angle A = $26^\circ 14'$, the side AC = 39, and the side AB = 53.

AC	39	1.5910646	R	10.
AB	53	1.7242759	Tan $8^\circ 39' 19''$	9.1823381
R		10.	Tan $\frac{1}{2}(B + C) 76^\circ 55'$	10.6326181
Tan $53^\circ 39' 9''$		10.1332113	Tan $\frac{1}{2}(B - C) 33^\circ 8' 50''$	9.8149562

The same result is obtained here, as by theorem II, p. 75.

To find the required *side* in this third case, by the theorems in section IV, it is necessary to find, in the first place, an *angle* opposite one of the given sides. But the required side may be obtained, in a different way, by the following proposition.

THEOREM B. *In a plane triangle, twice the product of any two sides, is to the difference between the sum of the squares of those sides, and the square of the third side, as radius to the cosine of the angle included between the two sides.*

In the triangle ABC, (Fig. 23.) whose sides are *a*, *b*, and *c*.

$$2bc : b^2 + c^2 - a^2 :: R : \cos A.$$

For in the right angled triangle ACD, $b : d :: R : \cos A$

Multiplying by $2c$, $2bc : 2dc :: R : \cos A$

But, by Euclid 13. 2, $2dc = b^2 + c^2 - a^2$

Therefore, $2bc : b^2 + c^2 - a^2 :: R : \cos A.$

The demonstration is the same, when the angle A is *obtuse*, as in the triangle ABC, (Fig. 24.) except that a^2 is *greater* than $b^2 + c^2$; (Euc. 12. 2.) so that the cosine of A is *negative*. See art. 194.

From this theorem, are derived expressions, both for the *sides* of a triangle, and for the cosines of the *angles*. Con-

verting the last proportion into an equation, and proceeding in the same manner with the other sides and angles, we have the following expressions;

$$\left\{ \begin{array}{l} \text{For the angles.} \\ \cos A = R \times \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B = R \times \frac{a^2 + c^2 - b^2}{2ac} \\ \cos C = R \times \frac{a^2 + b^2 - c^2}{2ab} \end{array} \right\} \left\{ \begin{array}{l} \text{For the sides.} \\ a = \sqrt{b^2 + c^2 - \frac{2bc \cos A}{R}} \\ b = \sqrt{a^2 + c^2 - \frac{2ac \cos B}{R}} \\ c = \sqrt{a^2 + b^2 - \frac{2ab \cos C}{R}} \end{array} \right\}$$

These formulæ are useful, in many trigonometrical investigations; but are not well adapted to logarithmic computation.

CASE IV.

When the *three sides* of a triangle are given, the angles may be found, by either of the following theorems; in which *a*, *b*, and *c*, are the sides, *A*, *B*, and *C*, the opposite angles, and *h*=half the sum of the sides.

$$\text{THEOREM C. } \left\{ \begin{array}{l} \sin A = \frac{2R}{bc} \sqrt{h(h-a)(h-b)(h-c)} \\ \sin B = \frac{2R}{ac} \sqrt{h(h-a)(h-b)(h-c)} \\ \sin C = \frac{2R}{ab} \sqrt{h(h-a)(h-b)(h-c)} \end{array} \right.$$

The quantities under the radical sign are the same in all the equations.

In the triangle ACD, (Fig. 23.)

$R : b :: \sin A : p$. Therefore, $\sin A \times b = R \times p$.

$$\text{But } p = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c} \quad (\text{Art. 221. p. 105.})$$

This, by the reductions in page 106, becomes

$$p = \frac{\sqrt{2h \times 2(h-a) \times 2(h-b) \times 2h-c}}{2c}$$

Substituting this value of p , and reducing,

$$\sin A = \frac{2R}{bc} \sqrt{h(h-a)(h-b)(h-c)}$$

The arithmetical calculations may be made, by adding the logarithms of the factors under the radical sign, dividing the sum by 2, and to the quotient, adding the logarithms of radius and 2, and the arithmetical complements of the logarithms of b and c . (Arts. 39, 47, 59.)

Ex. Given $a=134$, $b=108$, and $c=80$, to find A , B , and C .

For the angle A .

h	161	log.	2.2068259
$h-a$	27	log.	1.4313638
$h-b$	53	log.	1.7242759
$h-c$	81	log.	1.9084250

2)7.2709506

*3.6554753

$R \times 2$ log. 10.3010300

13.9365053

b 108 a. c. 7.9665762

c 80 a. c. 8.0969100

Sin A . 0.9999915

$A=89^\circ 38' 31''$

For the angle B .

a	134	a. c.	7.8728952
c	80	a. c.	8.0969100

Sin B . 9.9063105

$B=53^\circ 42' 9''$

For the angle C .

13.9365053

a 134 a. c. 7.8728952

b 108 a. c. 7.9665762

Sin C . 9.7759767

$C=36^\circ 39' 20''$

$$\text{THEOREM D. } \begin{cases} \sin \frac{1}{2}A = R \sqrt{\frac{(h-b)(h-c)}{bc}} \\ \sin \frac{1}{2}B = R \sqrt{\frac{(h-a)(h-c)}{ac}} \\ \sin \frac{1}{2}C = R \sqrt{\frac{(h-a)(h-b)}{ab}} \end{cases}$$

By art. 210, $2 \sin^2 \frac{1}{2}A = R^2 - R^2 \cos A$.

Substituting for $\cos A$, its value, as given in page 117,

$$2 \sin^2 \frac{1}{2}A = R^2 - R^2 \times \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{But } R^2 = R^2 \times \frac{2bc}{2bc} \text{ And } -R^2 \times \frac{b^2 + c^2 - a^2}{2bc} = R^2 \times \frac{a^2 - b^2 - c^2}{2bc}$$

* This is the logarithm of the area of the triangle. (Art. 222.)

$$\text{Therefore } 2\sin^2 \frac{1}{2}A = R^2 \times \frac{2bc + a^2 - b^2 - c^2}{2bc}$$

$$\text{But } 2bc + a^2 - b^2 - c^2 = a^2 - (b-c)^2 = (a+b-c)(a-b+c) \text{ (Alg. 235.)}$$

Putting then $h = \frac{1}{2}(a+b+c)$, reducing, and extracting,

$$\sin \frac{1}{2}A = R \sqrt{\frac{(h-b)(h-c)}{bc}}$$

Ex. Given a , b , and c , as before, to find A and B .

For the angle A .

$h-b$	53	1.7242759
$h-c$	81	1.9084850
$\frac{1}{2}$	108	a. c. 7.9665762
c	80	a. c. 8.0969100

$$\sin \frac{1}{2}A \quad \begin{array}{r} 2)19.6962471 \\ 9.8481235 \end{array}$$

$$A = 89^\circ 38' 31''$$

For the angle B .

$h-a$	27	1.4515638
$h-c$	81	1.9084850
a	154	a. c. 7.8728952
c	80	a. c. 8.0969100

$$\sin \frac{1}{2}B \quad \begin{array}{r} 2)19.3096540 \\ 9.6548270 \end{array}$$

$$B = 53^\circ 42' 9''$$

$$\text{THEOREM E. } \begin{cases} \cos \frac{1}{2}A = R \sqrt{\frac{h(h-a)}{bc}} \\ \cos \frac{1}{2}B = R \sqrt{\frac{h(h-b)}{ac}} \\ \cos \frac{1}{2}C = R \sqrt{\frac{h(h-c)}{ab}} \end{cases}$$

$$\text{By art. 210, } 2\cos^2 \frac{1}{2}A = R^2 + R \times \cos A$$

Substituting and reducing, as in the demonstration of the last theorem,

$$2\cos^2 \frac{1}{2}A = R^2 \times \frac{2bc + b^2 + c^2 - a^2}{2bc} = R^2 \times \frac{(b+c+a)(b+c-a)}{2bc}$$

Putting $h = \frac{1}{2}(a+b+c)$ reducing, and extracting,

$$\cos \frac{1}{2}A = R \sqrt{\frac{h(h-a)}{bc}}$$

α

Ex. Given the sides 134, 108, 80; to find B and C.

For the angle B.

h	161	2.2068259
$h-b$	53	1.7242759
a	134	a. c. 7.8728952
c	80	a. c. 8.0969100

$$\begin{array}{r} 2)19.9009070 \\ \hline \cos \frac{1}{2}B \\ \hline 9.9504535 \\ B=53^{\circ} 42' 9'' \end{array}$$

For the angle C.

h	161	2.2068259
$h-c$	81	1.9084850
a	134	a. c. 7.8728952
b	108	a. c. 7.9665762

$$\begin{array}{r} 2)19.9547823 \\ \hline \cos \frac{1}{2}C \\ \hline 9.9773911 \\ C=36^{\circ} 39' 20'' \end{array}$$

$$\text{THEOREM F.} \left\{ \begin{array}{l} \tan \frac{1}{2}A = R \sqrt{\frac{(h-b)(h-c)}{h(h-a)}} \\ \tan \frac{1}{2}B = R \sqrt{\frac{(h-a)(h-c)}{h(h-b)}} \\ \tan \frac{1}{2}C = R \sqrt{\frac{(h-a)(h-b)}{h(h-c)}} \end{array} \right.$$

The tangent is equal to the product of radius and the sine, divided by the cosine. (Art. 216.) By the two last theorems, then,

$$\tan \frac{1}{2}A = \frac{R \sin \frac{1}{2}a}{\cos \frac{1}{2}a} = R \sqrt{\frac{(h-b)(h-c)}{bc}} \div R \sqrt{\frac{h(h-a)}{bc}}$$

$$\text{That is, } \tan \frac{1}{2}A = R \sqrt{\frac{(h-b)(h-c)}{h(h-a)}}$$

Ex. Given the sides as before, to find A and C.

For the angle A.

$h-b$	53	1.7242759
$h-c$	81	1.9084850
$h-a$	27	a. c. 8.5686302
h	161	a. c. 7.7931741

$$\begin{array}{r} 2)19.9945712 \\ \hline \tan \frac{1}{2}A \\ \hline 9.9972856 \end{array}$$

$$A=89^{\circ} 38' 31''$$

For the angle C.

$h-a$	27	1.4313638
$h-b$	53	1.7242759
$h-c$	81	a. c. 8.0915150
h	161	a. c. 7.7931741

$$\begin{array}{r} 2)19.0403238 \\ \hline \tan \frac{1}{2}C \\ \hline 9.5201844 \end{array}$$

$$C=36^{\circ} 39' 20''$$

The three last theorems give the angle required, *without ambiguity*. For the *half* of any angle must be less than 90° .

Of these different methods of solution, each has its advantages in particular cases. It is expedient to find an angle, sometimes by its sine, sometimes by its cosine, and some-

times by its tangent. One reason for varying the mode of calculation, is the inequality in the increase of the sines, tangents, &c. in different parts of the quadrant.

The differences of the *sines* are greatest when the arcs are small, and least near 90° . An angle which is within 3 or 4 degrees of a right angle, can not be very accurately determined by its sine, as given in the tables. The same uncertainty attends the *cosine* of an angle which is very small.

The differences in the logarithmic *tangents* and *cotangents* are least at 45° , and increase towards each extremity of the quadrant. In no part of it, however, are they very small. In the tables which are carried to 7 places of decimals, the least difference for one second is 42. Any angle may be found, within one second, by its tangent, if tables are used which are calculated to seconds.

But the differences in the logarithmic sines and tangents, within a few minutes of the beginning of the quadrant, and in cosines and tangents within a few minutes of 90° , though they are very large, are yet too *unequal* to allow of an exact determination of their corresponding angles, by taking *proportional parts* of the differences. Very small angles may be accurately found, from their sines and tangents, by the rules given at the close of this note.

By the first of the four preceding theorems marked C, D, E, and F, the calculation is made for the *sine* of the *whole* angle; by the others, for the *sine*, *cosine*, or *tangent*, of *half* the angle. For finding an angle near 90° , each of the three last theorems is preferable to the first. In the example above, A would have been uncertain to several seconds, by theorem C, if the other two angles had not been determined also.

But for a very *small* angle, the first method has an advantage over the others. The third, by which the calculation is made for the *cosine* of half the required angle, is in this case the most defective of the four. The second will not answer well for an angle which is almost 180° . For the *half* of this is almost 90° ; and near 90° the differences of the *sines* are very small.

The following rules for finding the sine or tangent of a very small arc, and, on the other hand, for finding the arc from its sine or tangent, are taken from Dr. Maskelyne's Introduction to Taylor's Logarithms.

To find the logarithmic SINE of a very small arc.

From the sum of the constant quantity 4.6855749, and the logarithm of the given arc reduced to seconds and de

Q

subtract one third of the arithmetical complement of the logarithmic cosine.

To find the logarithmic TANGENT of a very small arc.

To the sum of the constant quantity 4.6855749, and the logarithm of the given arc reduced to seconds and decimals, add two thirds of the arithmetical complement of the logarithmic cosine.

To find a small arc from its logarithmic SINE.

To the sum of the constant quantity 5.3144251, and the given logarithmic sine, add one third of the arithmetical complement of the logarithmic cosine. The remainder diminished by 10, will be the logarithm of the number of seconds in the arc.

To find a small arc, from its logarithmic TANGENT.

From the sum of the constant quantity 5.3144251, and the given logarithmic tangent, subtract two thirds of the arithmetical complement of the logarithmic cosine. The remainder diminished by 10, will be the logarithm of the number of seconds in the arc.

For the demonstration of these rules, see Woodhouse's Trigonometry, p. 189.

NOTE I. p. 91.

Gunter's *Sliding Rule* is constructed upon the same principle as his scale, with the addition of a slider, which is so contrived as to answer the purpose of a pair of compasses, in working proportions, multiplying, dividing, &c. The lines on the *fixed part* are the same as on the scale. The *slider* contains two lines of numbers, a line of logarithmic sines, and a line of logarithmic tangents.

To *multiply* by this, bring 1 on the slider, against one of the factors on the fixed part; and against the other factor on the slider, will be the product on the fixed part. To *divide*, bring the divisor on the slider, against the dividend on the fixed part; and against 1 on the slider, will be the quotient on the fixed part. To work a *proportion*, bring the first term on the slider, against one of the middle terms on the fixed part; and against the other middle term on the slider, will be the fourth term on the fixed part. Or the first term may be taken on the fixed part; and then the fourth term will be found on the slider.

Another instrument frequently used in trigonometrical constructions, is

THE SECTOR.

This consists of two equal scales, moveable about a point as a centre. The lines which are drawn on it are of two kinds; some being parallel to the sides of the instrument, and others diverging from the central point, like the radii of a circle. The latter are called the *double lines*, as each is repeated upon the two scales. The *single lines* are of the same nature, and have the same use, as those which are put upon the common scale; as the lines of equal parts, of chords, of latitude, &c. on one face; and the logarithmic lines of numbers, of sines, and of tangents, on the other.

The *double lines* are

A line of <i>Lines</i> , or equal parts, marked	Lin. or L.
A line of <i>Chords</i> ,	Cho. or C.
A line of natural <i>Sines</i> ,	Sin. or S.
A line of natural <i>Tangents</i> to 45° ,	Tan. or T.
A line of tangents <i>above</i> 45° ,	Tan. or T.
A line of natural <i>Secants</i> ,	Sec. or S.
A line of <i>Polygons</i> ,	Pol. or P.

The double lines of *chords*, of *sines*, and of *tangents* to 45° , are all of the same radius; beginning at the central point, and terminating near the other extremity of each scale; the chords at 60° , the sines at 90° , and the tangents at 45° . (See art. 95.) The line of *lines* is also of the same length, containing ten equal parts which are numbered, and which are again subdivided. The radius of the lines of secants, and of tangents above 45° , is about one fourth of the length of the other lines. From the end of the radius, which for the secants is at 0, and for the tangents at 45° , these lines extend to between 70° and 80° . The line of polygons is numbered 4, 5, 6, &c. from the extremity of each scale, towards the centre.

The simple principle on which the utility of these several pairs of lines depends is this, that *the sides of similar triangles are proportional*. (Euc. 4. 6.) So that sines, tangents, &c. are furnished to *any radius*, within the extent of the opening of the two scales. Let AC and AC' (Fig. 40.) be any pair of lines on the sector, and AB and AB' equal por-

tions of these lines. As AC and AC' are equal, the triangle ACC' is isosceles, and similar to ABB'. Therefore,

$$AB : AC :: BB' : CC'.$$

Distances measured from the centre on either scale, as AB and AC, are called *lateral distances*. And the distances between corresponding points of the two scales, as BB' and CC', are called *transverse distances*.

Let AC and CC' be radii of two circles. Then if AB be the chord, sine, tangent, or secant, of any number of degrees in one; BB' will be the chord, sine, tangent, or secant, of the same number of degrees in the other. (Art. 119.) Thus, to find the *chord* of 30° , to a radius of four inches, open the sector so as to make the transverse distance from 60 to 60, on the lines of chords, four inches; and the distance from 30 to 30, on the same lines, will be the chord required. To find the *sine* of 28° , make the distance from 90 to 90, on the lines of sines, equal to radius; and the distance from 28 to 28 will be the sine. To find the *tangent* of 37° , make the distance from 45 to 45, on the lines of tangents, equal to radius; and the distance from 37 to 37 will be the tangent. In finding *secants*, the distance from 0 to 0 must be made radius. (Art. 201.)

To lay down an *angle* of 34° , describe a circle, of any convenient radius, open the sector so that the distance from 60 to 60 on the lines of chords shall be equal to this radius, and to the circle apply a chord equal to the distance from 34 to 34. (Art. 161.) For an angle above 60° , the chord of *half* the number of degrees may be taken, and applied *twice* on the arc, as in art. 161.

The line of *polygons* contains the chords of arcs of a circle which is divided into equal portions. Thus the distances from the centre of the sector to 4, 5, 6, and 7, are the chords of $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$ of a circle. The distance 6 is the radius. (Art. 95.) This line is used to make a regular polygon, or to inscribe one in a given circle. Thus, to make a *pentagon*, with the transverse distance from 6 to 6 for radius, describe a circle, and the distance from 5 to 5 will be the length of one of the sides of a pentagon inscribed in that circle.

The line of *lines* is used to divide a line into equal or proportional parts, to find fourth proportionals, &c. Thus, to divide a line into 7 equal parts, make the length of the given line the transverse distance from 7 to 7, and the distance from 1 to 1 will be one of the parts. To find $\frac{2}{3}$ of a line,

make the transverse distance from 5 to 5 equal to the given line; and the distance from 3 to 3 will be $\frac{3}{5}$ of it.

In working the *proportions in trigonometry* on the sector, the lengths of the sides of triangles are taken from the line of lines, and the degrees and minutes from the lines of sines, tangents, or secants. Thus in art. 135, ex. 1,

$$35 : R :: 26 : \sin 48^\circ.$$

To find the fourth term of this proportion by the sector, make the lateral distance 35 on the line of lines, a transverse distance from 90 to 90 on the lines of sines; then the lateral distance 26 on the line of lines, will be the transverse distance from 48 to 48 on the lines of sines.

For a more particular account of the construction and uses of the Sector, see Stone's edition of Bion on Mathematical Instruments, Hutton's Dictionary, and Robertson's Treatise on Mathematical Instruments.

NOTE K. p. 105.

Expressions for the *cotangents* may be obtained by putting

$$\cot = \frac{R^2}{\tan} \quad (\text{Art. 93.})$$

$$\text{Thus } \cot(a+b) = \frac{R^2}{\tan(a+b)} = \frac{R^2 - \tan a \tan b}{\tan a + \tan b} \quad (\text{Art. 218.})$$

Substituting $\frac{R^2}{\cot a}$ for $\tan a$, and $\frac{R^2}{\cot b}$ for $\tan b$,

$$\cot(a+b) = \frac{R^2 - \frac{R^2}{\cot a} \times \frac{R^2}{\cot b}}{\frac{R^2}{\cot a} + \frac{R^2}{\cot b}}$$

Multiplying both the numerator and denominator by $\cot a \cot b$, dividing by R^2 , and proceeding in the same manner, for $\cot(a-b)$ we have,

$$\text{I. } \cot(a+b) = \frac{\cot a \cot b - R^2}{\cot b + \cot a}$$

$$\text{II. } \cot(a-b) = \frac{\cot a \cot b + R^2}{\cot b - \cot a}$$

NOTE L. p. 109.

The error in supposing that arcs less than 1 minute are proportional to their sines, can not affect the first ten places of decimals. Let AB and AB' (Fig. 41.) each equal 1 minute. The tangents of these arcs BT and B'T are equal, as are also the sines BS and B'S. The arc BAB' is greater than BS+B'S, but less than BT+B'T. Therefore BA is greater than BS, but less than BT: that is, *the difference between the sine and the arc is less than the difference between the sine and the tangent.*

Now the sine of 1 minute is 0.000290888216

And the tangent of 1 minute is 0.000290888204

The difference is 0.000000000012

The difference between the sine and the arc of 1 minute is less than this; and the error in supposing that the sines of 1', and of 0° 52' 44" 3''' 45'''' are proportional to their arcs, as in art. 223, is still less.

NOTE M. p. 110.

There are various ways in which sines and cosines may be more *expeditiously* calculated, than by the method which is given here. But as we are already supplied with accurate trigonometrical tables, the computation of the canon is, to the great body of our students, a subject of speculation, rather than of practical utility. Those who wish to enter into a minute examination of it, will of course consult the treatises in which it is particularly considered.

There are also numerous formulæ of *verification*, which are used to detect the errors with which any part of the calculation is liable to be affected. For these, see Legendre's and Woodhouse's Trigonometry, Lacroix's Differential Calculus, and particularly Euler's Analysis of Infinites.

A TABLE OF

NATURAL SINES AND TANGENTS;

TO EVERY TEN MINUTES OF A DEGREE.

IF the given angle is less than 45° , look for the title of the column, at the *top* of the page; and for the degrees and minutes, on the *left*. But if the angle is between 45° and 90° , look for the title of the column, at the *bottom*; and for the degrees and minutes, on the *right*.

NATURAL SINES AND TANGENTS.

D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
0° 0'	0.0000000	0.0000000	Infinite	1.0000000	90° 0'
10	0029089	0029089	343.77371	0.9999958	50
20	0058177	0058178	171.88540	9999831	40
30	0087265	0087269	114.58865	9999619	30
40	0116353	0116361	85.939791	9999323	20
0° 50'	0145439	0145454	68.750087	9998942	89° 10'
1° 0'	0.0174524	0.0174551	57.289962	0.9998477	89° 0'
10	0203608	0203650	49.103881	9997927	50
20	0232690	0232753	42.964077	9997292	40
30	0261769	0261859	38.188459	9996573	30
40	0290847	0290970	34.367771	9995770	20
1° 50'	0319922	0320086	31.241577	9994881	88° 10'
2° 0'	0.0348995	0.0349208	28.636253	0.9993908	88° 0'
10	0378065	0378335	26.431600	9992851	50
20	0407131	0407469	24.541758	9991709	40
30	0436194	0436609	22.903766	9990482	30
40	0465253	0465757	21.470401	9989171	20
2° 50'	0494308	0494913	20.205553	9987775	87° 10'
3° 0'	0.0523360	0.0524078	19.081137	0.9986295	87° 0'
10	0552406	0553251	18.074977	9984731	50
20	0581448	0582434	17.169337	9983082	40
30	0610485	0611626	16.349855	9981348	30
40	0639517	0640829	15.604784	9979530	20
3° 50'	0668544	0670043	14.924417	9977627	86° 10'
4° 0'	0.0697565	0.0699268	14.300666	0.9975641	86° 0'
10	0726580	0728505	13.726738	9973569	50
20	0755589	0757755	13.196883	9971413	40
30	0784591	0787017	12.706205	9969173	30
40	0813587	0816293	12.250505	9966849	20
4° 50'	0842576	0845583	11.826167	9964440	85° 10'
5° 0'	0.0871557	0.0874887	11.430052	0.9961947	85° 0'
10	0900532	0904206	11.059431	9959370	50
20	0929498	0933540	10.711913	9956708	40
30	0958458	0962890	10.385397	9953962	30
40	0987408	0992257	10.078031	9951132	20
5° 50'	1016351	1021641	9.7881732	9948217	84° 10'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

NATURAL SINES AND TANGENTS.

D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
6° 0'	0.1045285	0.1051042	9.5143645	0.9945219	84° 0'
10	1074210	1080462	9.2553035	9942130	50
20	1103126	1109899	9.0098261	9938969	40
30	1132032	1139356	8.7768874	9935719	30
40	1160929	1168832	8.5555468	9932384	20
6° 50'	1189816	1198329	8.3449558	9928065	83° 10'
7° 0'	0.1218693	0.1227846	8.1443464	0.9925462	83° 0'
10	1247560	1257384	7.9530224	9921874	50
20	1276416	1286943	7.7703506	9918204	40
30	1305262	1316525	7.5957541	9914449	30
40	1334096	1346129	7.4287064	9910610	20
7° 50'	1362919	1375757	7.2687255	9906687	82° 10'
8° 0'	0.1391731	0.1404085	7.1153697	0.9902681	82° 0'
10	1420531	1435084	6.9682335	9898590	50
20	1449319	1464784	6.8269437	9894416	40
30	1478094	1494510	6.6911562	9890159	30
40	1506857	1524262	6.5605538	9885817	20
8° 50'	1535607	1554040	6.4348428	9881392	81° 10'
9° 0'	0.1564345	0.1583844	6.3137515	0.9876883	81° 0'
10	1593069	1613677	6.1970279	9872291	50
20	1621779	1643537	6.0844381	9867615	40
30	1650476	1673426	5.9757644	9862856	30
40	1679159	1703344	5.8708042	9858013	20
9° 50'	1707828	1733292	5.7693688	9853087	80° 10'
10° 0'	0.1736462	0.1763270	5.6712818	0.9848078	80° 0'
10	1765121	1793279	5.5763786	9842985	50
20	1793746	1823319	5.4845052	9837808	40
30	1822355	1853390	5.3955172	9832549	30
40	1850949	1883495	5.3092793	9827206	20
10° 50'	1879528	1913632	5.2256647	9821781	79° 10'
11° 0'	0.1908090	0.1943803	5.1445540	0.9816272	79° 0'
10	1936636	1974008	5.0656352	9810680	50
20	1965166	2004248	4.9894027	9805005	40
30	1993679	2034523	4.9151570	9799247	30
40	2022176	2064834	4.8430045	9793406	20
11° 50'	2050655	2095181	4.7728568	9787483	78° 10'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

NATURAL SINES AND TANGENTS.

D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
12° 0'	0.2079117	0.2125366	4.7046301	0.9781476	78° 0'
10	2107561	2155988	4.6382457	9775387	50
20	2135988	2186448	4.5736287	9769215	40
30	2164396	2216947	4.5107085	9762960	30
40	2192786	2247485	4.4494181	9756623	20
12° 50'	2221158	2278063	4.3896940	9750203	77° 10'
13° 0'	0.2249511	0.2308682	4.3314759	0.9745701	77° 0'
10	2277844	2339342	4.2747066	9737116	50
20	2306159	2370044	4.2193318	9730449	40
30	2334454	2400788	4.1652998	9723699	30
40	2362729	2431575	4.1125614	9716867	20
13° 50'	2390984	2462405	4.0610700	9709953	76° 10'
14° 0'	0.2419219	0.2493280	4.0107809	0.9702957	76° 0'
10	2447433	2524200	3.9616518	9695879	50
20	2475627	2555165	3.9136420	9688719	40
30	2503800	2586176	3.8667131	9681476	30
40	2531952	2617234	3.8208281	9674152	20
14° 50'	2560082	2648339	3.7759519	9666746	75° 10'
15° 0'	0.2588190	0.2679492	3.7320508	0.9659258	75° 0'
10	2616277	2710694	3.6890927	9651689	50
20	2644342	2741945	3.6470467	9644037	40
30	2672384	2773245	3.6058835	9636305	30
40	2700403	2804597	3.5655749	9628490	20
15° 50'	2728400	2835999	3.5260938	9620594	74° 10'
16° 0'	0.2756374	0.2867454	3.4874144	0.9612617	74° 0'
10	2784324	2898961	3.4495120	9604558	50
20	2812251	2930521	3.4123626	9596418	40
30	2840153	2962135	3.3759434	9588197	30
40	2868032	2993803	3.3402326	9579895	20
16° 50'	2895887	3025527	3.3052091	9571512	73° 10'
17° 0'	0.2923717	0.3057307	3.2708526	0.9563048	73° 0'
10	2951522	3089143	3.2371438	9554502	50
20	2979303	3121036	3.2040638	9545876	40
30	3007058	3152988	3.1715948	9537170	30
40	3034788	3184998	3.1397194	9528382	20
17° 50'	3062492	3217067	3.1084210	9519514	72° 10'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

NATURAL SINES AND TANGENTS.

D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
18° 0'	0.3090170	0.3249197	3.0776835	0.9510565	72° 0'
10	3117822	3281387	3.0474915	9501536	50
20	3145448	3313639	3.0178301	9492426	40
30	3173047	3345953	2.9886850	9483257	30
40	3200619	3378330	2.9600422	9473966	20
18° 50'	3228164	3410771	2.9318885	9464616	71° 10'
19° 0'	0.3253682	0.3443276	2.9042109	0.9455186	71° 0'
10	3283172	3475846	2.8769970	9445675	50
20	3310634	3508483	2.8502349	9436085	40
30	3338069	3541186	2.8239129	9426415	30
40	3365475	3573956	2.7980198	9416665	20
19° 50'	3392852	3606795	2.7725448	9406835	70° 10'
20° 0'	0.3420201	0.3639702	2.7474774	0.9396926	70° 0'
10	3447521	3672680	2.7228076	9386938	50
20	3474812	3705728	2.6985254	9376869	40
30	3502074	3738847	2.6746215	9366722	30
40	3529306	3772038	2.6510867	9356495	20
20° 50'	3556508	3805302	2.6279121	9346189	69° 10'
21° 0'	0.3583679	0.3838640	2.6050891	0.9335804	69° 0'
10	3610821	3872053	2.5826094	9325340	50
20	3637932	3905541	2.5604645	9314797	40
30	3665012	3939105	2.5386479	9304176	30
40	3692061	3972746	2.5171507	9293475	20
21° 50'	3719079	4006465	2.4959661	9282696	68° 10'
22° 0'	0.3746066	0.4040262	2.4750869	0.9271839	68° 0'
10	3773021	4074139	2.4545061	9260902	50
20	3799944	4108097	2.4342172	9249888	40
30	3826834	4142136	2.4142136	9238755	30
40	3853693	4176257	2.3944889	9227624	20
22° 50'	3880518	4210460	2.3750372	9216375	67° 10'
23° 0'	0.3907311	0.4244748	2.3558524	0.9205049	67° 0'
10	3934071	4279121	2.3369287	9193644	50
20	3960798	4313579	2.3182606	9182161	40
30	3987491	4348124	2.2998425	9170601	30
40	4014150	4382756	2.2816693	9158963	20
23° 50'	4040775	4417477	2.2637357	9147247	66° 10'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

NATURAL SINES AND TANGENTS.

D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
24° 0'	0.4067366	0.4452287	2.2460368	0.9135455	66° 0'
10	4093923	4487187	2.2285676	9123584	50
20	4120445	4522179	2.2113234	9111637	40
30	4146932	4557263	2.1942997	9099613	30
40	4173385	4592439	2.1774920	9087511	20
24° 50'	4199801	4627710	2.1608958	9075333	65° 10'
25° 0'	0.4226183	0.4663077	2.1445069	0.9063078	65° 0'
10	4252528	4698539	2.1283213	9050746	50
20	4278838	4734098	2.1123348	9038338	40
30	4305111	4769755	2.0965436	9025853	30
40	4331348	4805512	2.0809438	9013292	20
25° 50'	4357548	4841368	2.0655318	9000654	64° 10'
26° 0'	0.4383711	0.4877326	2.0503038	0.8987940	64° 0'
10	4409838	4913386	2.0352565	8975151	50
20	4435927	4949549	2.0203862	8962285	40
30	4461978	4985816	2.0056897	8949344	30
40	4487992	5022189	1.9911637	8936326	20
26° 50'	4513967	5058668	1.9768050	8923234	63° 10'
27° 0'	0.4539905	0.5095254	1.9626105	0.8910065	63° 0'
10	4565804	5131950	1.9485772	8896822	50
20	4591665	5168755	1.9347020	8883503	40
30	4617486	5205671	1.9209821	8870108	30
40	4643269	5242698	1.9074147	8856639	20
27° 50'	4669012	5279839	1.8939971	8843095	62° 10'
28° 0'	0.4694716	0.5317094	1.8807265	0.8829476	62° 0'
10	4720380	5354465	1.8676003	8815782	50
20	4746004	5391952	1.8546159	8802014	40
30	4771588	5429557	1.8417709	8788171	30
40	4797131	5467281	1.8290628	8774254	20
28° 50'	4822634	5505125	1.8164892	8760263	61° 10'
29° 0'	0.4848096	0.5543091	1.8040478	0.8746197	61° 0'
10	4873517	5581179	1.7917362	8732058	50
20	4898897	5619391	1.7795524	8717844	40
30	4924236	5657728	1.7674940	8703557	30
40	4949532	5696191	1.7555590	8689196	20
29° 50'	4974787	5734783	1.7437453	8674762	60° 10'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

NATURAL SINES AND TANGENTS.

D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
30° 0'	0.5000000	0.5773503	1.7320508	0.8660254	60° 0'
10	5025170	5812353	1.7204736	8645673	50
20	5050298	5851335	1.7090116	8631019	40
30	5075384	5890450	1.6976631	8616292	30
40	5100426	5929699	1.6864261	8601491	20
30° 50'	5125425	5969084	1.6752988	8586619	59° 10'
31° 0'	0.5150381	0.6008606	1.6642795	0.8571673	59° 0'
10	5175293	6048266	1.6533663	8556655	50
20	5200161	6088067	1.6425576	8541564	40
30	5224986	6128008	1.6318517	8526402	30
40	5249766	6168092	1.6212469	8511167	20
31° 50'	5274502	6208320	1.6107417	8495860	58° 10'
32° 0'	0.5299193	0.6248694	1.6003345	0.8480481	58° 0'
10	5323839	6289214	1.5900238	8465030	50
20	5348440	6329883	1.5798079	8449508	40
30	5372996	6370703	1.5696856	8433914	30
40	5397507	6411673	1.5596552	8418249	20
32° 50'	5421971	6452797	1.5497155	8402513	57° 10'
33° 0'	0.5446390	0.6494076	1.5398650	0.8386706	57° 0'
10	5470763	6535511	1.5301023	8370827	50
20	5495090	6577103	1.5204261	8354878	40
30	5519370	6618856	1.5108352	8338858	30
40	5543603	6660769	1.5013282	8322768	20
33° 50'	5567790	6702845	1.4919039	8306607	56° 10'
34° 0'	0.5591929	0.6745085	1.4825610	0.8290376	56° 0'
10	5616021	6787492	1.4732983	8274074	50
20	5640066	6830066	1.4641147	8257703	40
30	5664062	6872810	1.4550090	8241262	30
40	5688011	6915725	1.4459801	8224751	20
34° 50'	5711912	6958613	1.4370268	8208170	55° 10'
35° 0'	0.5735764	0.7002075	1.4281480	0.8191520	55° 0'
10	5759568	7045515	1.4193427	8174801	50
20	5783323	7089133	1.4106098	8158013	40
30	5807030	7132931	1.4019483	8141155	30
40	5830687	7176911	1.3933571	8124229	20
35° 50'	5854294	7221075	1.3848353	8107234	54° 10'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

NATURAL SINES AND TANGENTS.

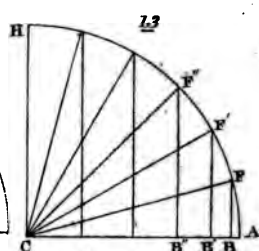
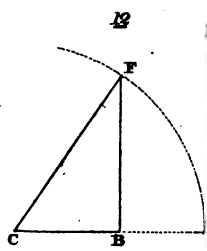
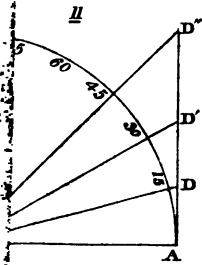
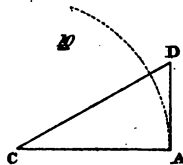
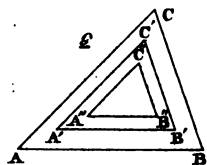
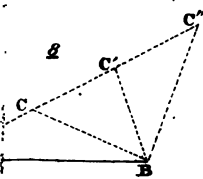
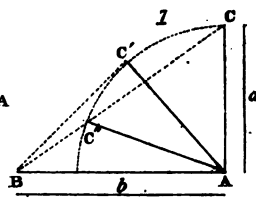
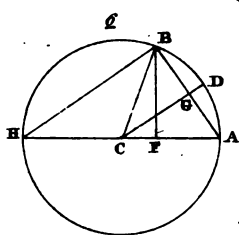
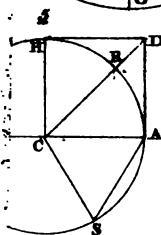
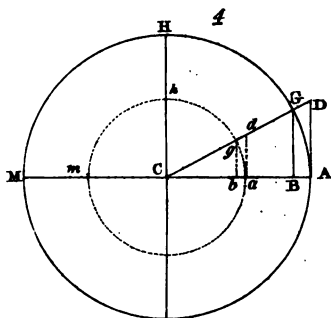
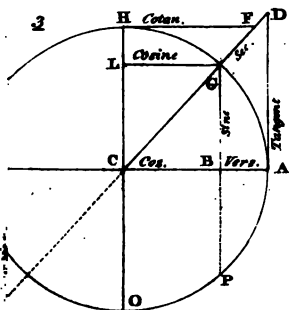
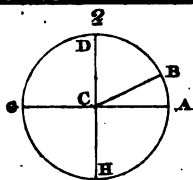
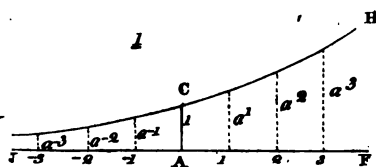
D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
36° 0'	0.5877853	0.7265425	1.3763819	0.8090170	54° 0'
10	5901361	7309963	1.3679959	8073038	50
20	5924819	7354691	1.3596764	8055837	40
30	5948228	7399611	1.3514224	8038569	30
40	5971586	7444724	1.3432331	8021232	20
36° 50'	5994893	7490033	1.3351075	8003827	53° 10'
37° 0'	0.6018150	0.7535541	1.3270448	0.7986355	53° 0'
10	6041356	7581248	1.3190441	7968815	50
20	6064511	7627157	1.3111046	7951208	40
30	6087614	7673270	1.3032254	7933533	30
40	6110666	7719589	1.2954057	7915792	20
37° 50'	6133666	7766118	1.2876447	7897983	52° 10'
38° 0'	0.6156615	0.7812656	1.2799416	0.7880108	52° 0'
10	6179511	7859808	1.2722957	7862165	50
20	6202355	7906975	1.2647062	7844157	40
30	6225146	7954359	1.2571723	7826082	30
40	6247885	8001963	1.2496933	7807940	20
38° 50'	6270571	8049790	1.2422685	7789733	51° 10'
39° 0'	0.6293204	0.8097840	1.2348972	0.7771460	51° 0'
10	6315784	8146118	1.2275786	7753121	50
20	6338310	8194625	1.2203121	7734716	40
30	6360782	8243364	1.2130970	7716246	30
40	6383201	8292337	1.2059327	7697710	20
39° 50'	6405566	8341547	1.1988184	7679110	50° 10'
40° 0'	0.6427876	0.8390996	1.1917536	0.7660444	50° 0'
10	6450132	8440688	1.1847376	7641714	50
20	6472334	8490624	1.1777698	7622919	40
30	6494480	8540807	1.1708496	7604060	30
40	6516572	8591240	1.1639763	7585136	20
40° 50'	6538609	8641926	1.1571495	7566148	49° 10'
41° 0'	0.6560590	0.8692867	1.1503684	0.7547096	49° 0'
10	6582516	8744067	1.1436326	7527980	50
20	6604386	8795528	1.1369414	7508800	40
30	6626200	8847253	1.1302944	7489557	30
40	6647959	8899244	1.1236909	7470251	20
41° 50'	6669661	8951506	1.1171305	7450881	48° 10'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

NATURAL SINES AND TANGENTS.

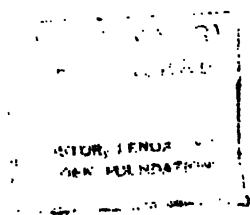
D. M.	Sine	Tangent	Cotangent	Cosine	D. M.
42° 0'	0.6691306	0.9004040	1.1106125	0.7431448	48° 0'
10	6712895	9056851	1.1041365	7411953	50
20	6734427	9109940	1.0977020	7392394	40
30	6755902	9163312	1.0913085	7372773	30
40	6777320	9216969	1.0849554	7353090	20
42° 50'	6798681	9270914	1.0786423	7333345	47° 10'
43° 0'	0.6819984	0.9325151	1.0723687	0.7313537	47° 0'
10	6841229	9379683	1.0661341	7293668	50
20	6862416	9434513	1.0599381	7273736	40
30	6883546	9489646	1.0537801	7253744	30
40	6904617	9545083	1.0476598	7233690	20
43° 50'	6925630	9600829	1.0415767	7213574	46° 10'
44° 0'	0.6946584	0.9656888	1.0355303	0.7193398	46° 0'
10	6967479	9713262	1.0295203	7173161	50
20	6988315	9769956	1.0235461	7152863	40
30	7009093	9826973	1.0176074	7132504	30
40	7029811	9884316	1.0117038	7112086	20
44° 50'	7050469	9941991	1.0058348	7091607	45° 10'
45° 0'	0.7071068	1.0000000	1.0000000	0.7071068	45° 0'
D. M.	Cosine	Cotangent	Tangent	Sine	D. M.

The *Secants* and *Cosecants*, which are not inserted in this table, may be easily supplied. If 1 be divided by the cosine of an arc, the quotient will be the secant of that arc. (Art. 228.) And if 1 be divided by the sine, the quotient will be the cosecant.

P.I.



N. Jacolin, S.M.N.

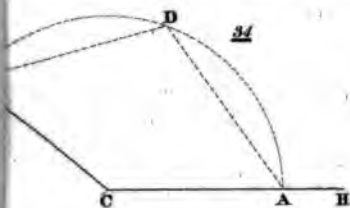


TRIGONOMETRY.

PL. III.

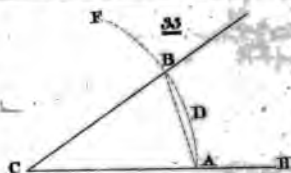
32

C D E B

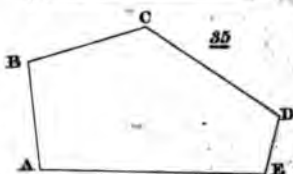


34

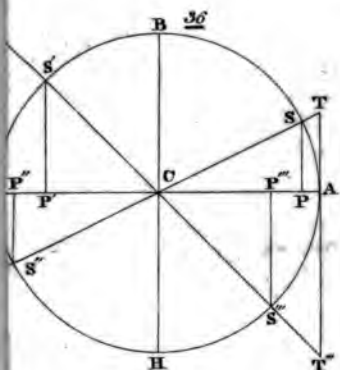
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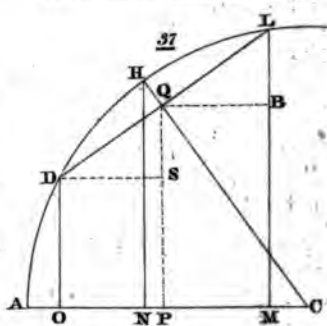
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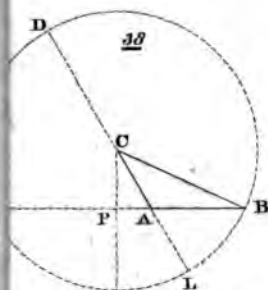
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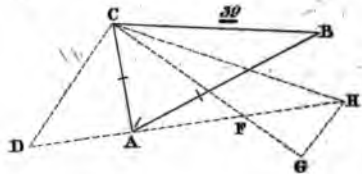
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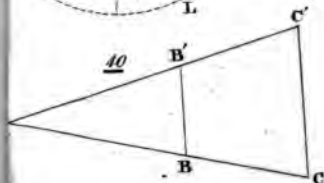
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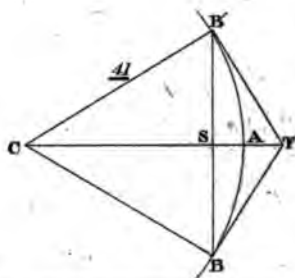
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41



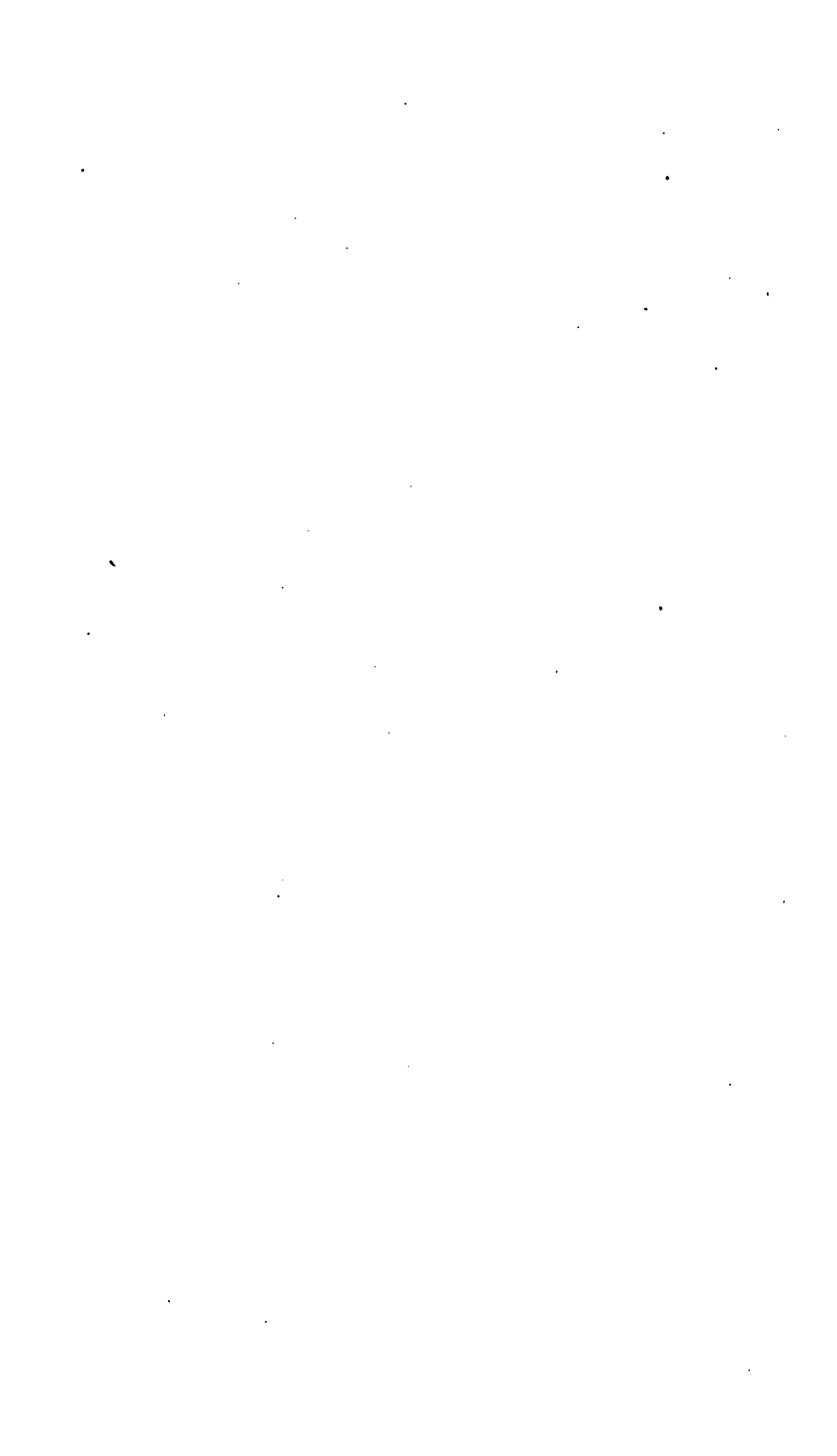
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